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The concept of monodromy for linear problems and its applications

T E S I S

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Misael Avendaño Camacho

Director de Tesis: Dr. Yuri Vorobev

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SINODALES

Dr. Yuri Vorobev
Universidad de Sonora, Hermosillo, México.

Dr. Ruben Flores Espinoza
Universidad de Sonora, Hermosillo, México.

Dr. Rodrigo Gonzalez Gonzalez
Universidad de Sonora, Hermosillo, México.

Dr. Vladimir Danilov
Moscow Technical University of Comunications and Informatics, Moscow, Rusia.

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Introduction

The notion of a monodromy matrix (operator) naturally appears under the study of linear systems with periodic coefficients. This notion gives rise to the well known result [4, 17] on the reducibility of linear periodic systems (Floquet's Theorem) which says that the monodromy matrix contains the complete information about a given system.

The goal of the present work is to develop an unified viewpoint of monodromy for linear systems on Lie algebras with quasi-periodic and decreasing boundary conditions. The quasi-periodic case is a natural generalization of the periodic case, and the decreasing case can be interpreted as the "limiting" periodic case (the period tends to infinity). Such a class of linear problems arise in the integrability theory of nonlinear partial differential equations in the framework of the so-called inverse scattering method [8, 11].

If a nonlinear partial differential equations can be represented as the consistence condition for two linear problems (called the L-A pair), then the inverse scattering method allow us reconstruct a wide class of solutions of the nonlinear equations from the corresponding "spectral data". This leads to the study of the zero curvature equation and Lax equation [8, 11]. The main point here is the study of the analytic properties of the monodromy matrix depending on a spectral parameter and then the time-evolution of the spectral data.

As an application of the general concept of monodromy, we consider a class of linear problems associated with integrability of nonlinear Schrödinger equation (NLS equation) [8, 11].

The present work is organized as follows:

In chapter 1, we give a definition and describe general properties of the monodromy for linear systems on general matrix Lie algebras complemented with the boundary conditions of the following three types : periodic, quasi-periodic and decreasing. We show that the natural definition of monodromy for linear systems with periodic conditions can be extended to quasi-periodic and decreasing cases. Moreover, we study the zero curvature equation and its coupled linear system and establish important results on the time evolution of monodromy and conservation laws.

The Chapter 2 begins with some applications of the general results, obtained in Chapter 1, to linear systemss on the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, possessing the involution property. Such a class of linear problems plays an important role in the integrability theory for nonlinear evolution equations. We derive also new properties of the fundamental solution and monodromy matrix in the case of $\mathfrak{sl}(2,\mathbb{C})$, which allow us introduce the concept of transitions coefficients. Interpreting the linear system with boundary condition as an spectral problem, we arrive to the notion of spectral data that are provided by the monodromy matrix. Then, we study the time evolution of the spectral data for linear systems on $\mathfrak{sl}(2,\mathbb{C})$ in the decreasing case. The principal results of this Chapter are formulated in Theorem 2.3.1, Proposition 2.4.3, Proposition 2.4.5, and Proposition 2.5.2.

The Chapter 3 is devoted to the inverse problem for the linear system in $\mathfrak{sl}(2,\mathbb{C})$ with decreasing coefficients. The question here is to reconstruct the linear system from its spectral data. To do that, the idea is to reformulate this question in terms of the Riemann-Hilbert problem which is known in the context of the complex analysis and the theory of integral equations [1], The main results are presented in Theorem 3.1.1 and Theorem 3.1.2. Finally, we give some applications of these results to linear systems coming from the nonlinear Schrödinger equation.

Contents

Chapter 1

Monodromy of Linear Systems with Boundary Conditions

The goal of this chapter is to give a definition of the monodromy for linear system on general matrix Lie algebras complemented with the boundary conditions of the following three types: the periodic, quasi-periodic and decreasing.

The notion of monodromy naturally appears in the case of linear systems with periodic conditions. We show that the notion of monodromy can be also extended to the quasi-periodic and decreasing cases. Moreover, we discuss some properties of the monodromy for these linear boundary problem.

At the end of the chapter, we deal with the zero curvature equation and its coupled linear system. We present some important results on the time evolution of the monodromy of linear systems admitting a zero curvature representation. .

1.1 Fundamental Solutions

Let $\mathcal{V}$ be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Denote by $\mathfrak{gl}(\mathcal{V})$ the Lie algebra of all the linear transformation of $\mathcal{V}$ and by $\mathbf{GL}(\mathcal{V})$ the **general linear group** consisting of all invertible linear transformation.

Given a C^∞ linear function $\mathbb{R} \ni x \mapsto \mathbf{U}(x) \in \mathfrak{gl}(\mathcal{V})$, consider the follows linear system

$$\frac{\mathrm{d}\mathrm{f}}{\mathrm{d}x} = \mathbf{U}(x)\mathrm{f}, \quad (\mathrm{f} = \mathrm{f}(x) \in \mathcal{V}) \tag{1.1}$$

We shall assume that $\mathbf{U}$ is bounded in $\mathbb{R}$ with respect to some norm on $\mathfrak{gl}(\mathcal{V})$

$$\|\mathbf{U}(x)\| < \infty, \quad \text{on} \quad \mathbb{R}. \tag{1.2}$$

Then, as is well known, [14, 15], there exists the **fundamental solution** of (1.1), i.e., a function $\mathbb{R}^2 \ni (x, y) \mapsto \mathbf{F}(x, y) \in \mathfrak{gl}(\mathcal{V})$ satisfying the Cauchy Problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}(x,y) = \mathbf{U}(x)\mathbf{F}(x,y), \tag{1.3}$$

$$\mathbf{F}(x,y)|_{x=y} = \mathbf{I}, \tag{1.4}$$

for every $y \in \mathbb{R}$. The solution of (1.1) with initial data $\mathbf{f}|_{x=y} = f^0$ is given by

$$\mathbf{f}(x) = \mathbf{F}(x,y)f^0. \tag{1.5}$$

For a fixed $y \in \mathbb{R}$.

Proposition 1.1.1. *The fundamental solution $\mathbf{F}(x,y)$ is differentiable in x, y and has the properties:*

(i) Non-degeneracy,

$$\det \mathbf{F}(x,y) \neq 0., \tag{1.6}$$

and hence $\mathbf{F}(x,y) \in \mathbf{GL}(\mathcal{V})$.

(ii) The transition property

$$\mathbf{F}(x,z)\mathbf{F}(z,y) = \mathbf{F}(x,y), \tag{1.7}$$

for all x, y z.

(iii) The inverse of the fundamental solution is given by

$$\mathbf{F}^{-1}(x,y) = \mathbf{F}(y,x),$$

and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathbf{F}(x,y) = -\mathbf{F}(x,y)\mathbf{U}(y).$$

Proof. (i) Let, $s \in \mathbb{R}$ be a fixed point. The fundamental solution $\mathbf{F}(x,y)$ can be seen as an integrable curve on the differentiable manifold $\mathfrak{gl}(V)$, so the fundamental solution $\mathbf{F}(x,y)$ is a continuous map that joints the identity element $\mathbf{I}$, with the point $\mathbf{F}(s,y)$. Therefore, for each x the fundamental solution $\mathbf{F}(x,y)$ is in the connected component of the identity element $\mathfrak{gl}^0(V)$. Since the determinant of a matrix is a continuous function

$$\mathbf{GL}^0(V) = \{X \in \mathbf{GL}(V) | \det X > 0\},$$

and therefore, $\mathbf{F}(x,y)$ is in $\mathbf{GL}(V)$.

(ii) Fixed some points y and z. The matrix function $\mathbf{G}(x,z) = \mathbf{F}(x,y)$ is the solution of the Cauchy problem (1.3) satisfying the boundary condition $\mathbf{G}|_{x=z} = \mathbf{G}_0(z) = \mathbf{F}(z,y)$. Since the matrix function $\mathbf{H}(x,y) = \mathbf{F}(x,z)\mathbf{F}(z,y)$ satisfies the same Cauchy problem, the uniqueness implies $\mathbf{F}(x,z)\mathbf{F}(z,y) = \mathbf{F}(x,y)$.

(iii) By *(ii)* we have

$$\mathbf{F}(x,y)\mathbf{F}(y,x) = \mathbf{F}(x,x) = \mathbf{I}$$

then $\mathbf{F}^{-1}(x,y) = \mathbf{F}(y,x)$.

iv) By (ii), we know that $\mathbf{F}(x,y)\mathbf{F}(y,x) = \mathbf{I}$. From this, we can derive

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{F}(x,y)}{\mathrm{dy}} &= -\mathbf{F}(x,y)\frac{\mathrm{d}\mathbf{F}(y,x)}{\mathrm{dy}}\mathbf{F}(x,y) \\
&= -\mathbf{F}(x,y)\mathbf{U}(y)\mathbf{F}(y,x)\mathbf{F}(x,y) \\
&= -\mathbf{F}(x,y)\mathbf{U}(y)
\end{aligned}$$

□

1.1.1 Linear Systems on Matrix Lie Algebras

Consider linear system (1.1), let $\mathfrak{g}$ be a subalgebra of $\mathbf{gl}(n,\mathbb{F})$.

Assuming that the coefficient $\mathbf{U}(x)$ takes values in $\mathfrak{g}$,we denote by $\mathbf{GL}(n,\mathbb{F})$ the Lie group of $n \times n$ invertible matrices. For each X a $n \times n$ real or complex matrix, the exponential of X, denoted by e^X or $\exp X$, is defined by the power series

$$\mathrm{e}^X = \sum_{i=0}^{\infty} \frac{X^m}{m!}. \tag{1.8}$$

The map $t \mapsto \exp tX$ is a smooth curve in $\mathbf{gl}(n,\mathbb{F})$.

Let $G \subset \mathbf{GL}(n)$ be a matrix Lie subgroup. The Lie algebra of G, denoted by $\mathfrak{g}$, is the set of matrices X such that e^{tX} is in G for all real t,

$$\mathfrak{g} := \{X \in \mathbf{gl}(n,\mathbb{F}) | \exp(tX) \in G, \forall t\}. \tag{1.9}$$

In fact, $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{gl}(n,\mathbb{F})$.

If G is a Lie group, the component connected of the element identity in G is a subgroup of G denoted by G^0. According to [7], we have the following properties:

- G_0 is pathwise connected,
- G_0 is open and closed subset in G,

- G_0 is a normal subgroup of G.

Proposition 1.1.2. *Let* $\mathfrak{g}$ *be a subalgebra of* $\mathbf{gl}(n,\mathbb{F})$. *Fix a smooth map* $\mathbf{U}(x) \in \mathfrak{g}$ *such satisfy (1.2). If* $\mathbf{F}(x,y)$ *is the fundamental solution of the linear system (1.1) on* $\mathfrak{g}$, *then* $\mathbf{F}(x,y)$ *lies in the connected component of the identity element of* $\mathbf{GL}(n)$.

Proof. The fundamental solution $\mathbf{F}(x,y)$ is a continuous curve which joints the identity matrix $\mathbf{I}$ with $\mathbf{F}(x,y)$. Since $\mathbf{GL}^0(n)$ is pathwise connected, it follows that $\mathbf{F}(x,y)$ must belongs to the connected component of the identity. □

Change of Coordinates

Assume that $\mathbf{f}(x)$ is a solution of the linear system (1.1) in the Lie algebra $\mathfrak{g}$, with initial data $\mathbf{f}\,|_{x=y} = \mathbf{f}^0$. Let $\mathbf{B}(x)$ be a C^∞ matrix map in $\mathbf{GL}(n,\mathbb{F})$. The change of coordinates $\widetilde{\mathbf{f}}(x) = \mathbf{B}(x)\mathbf{f}(x)$ of the linear system (1.1) leads to the linear system of the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{\mathbf{f}}(x) = \widetilde{\mathbf{U}}(x)\widetilde{\mathbf{f}}(x), \tag{1.10}$$

$$\widetilde{\mathbf{f}}(x)\,|_{x=y} = \widetilde{\mathbf{f}}^0. \tag{1.11}$$

The matrix coefficient $\widetilde{\mathbf{U}}(x)$ is given by

$$\widetilde{\mathbf{U}} = \frac{\partial \mathbf{B}}{\partial x}\mathbf{B}^{-1} + \mathbf{B}\mathbf{U}\mathbf{B}^{-1}.$$

Proposition 1.1.3. *Let* $\mathbf{F}(x,y)$ *and* $\widetilde{\mathbf{F}}(x,y)$ *be the fundamental solutions of linear systems (1.10)-(1.11) and (1.1) on* $\mathfrak{g}$, *respectively. Then*

$$\tilde{\mathbf{F}}(x,y) = \mathbf{B}(x)\mathbf{F}(x,y)\mathbf{B}^{-1}(x). \tag{1.12}$$

Proof. Since $\widetilde{\mathbf{F}}(x,y)$ is the fundamental solution of (1.10), we have

$$\begin{aligned}\widetilde{\mathbf{F}}(x,y)\widetilde{\mathbf{f}}^0 &= \widetilde{\mathbf{f}}(x)\\ &= \mathbf{B}(x)\mathbf{f}(x)\\ &= \mathbf{B}(x)\mathbf{F}(x,y)\mathbf{f}^0\\ &= \mathbf{B}(x)\mathbf{F}(x,y)\mathbf{B}^{-1}(x)\widetilde{\mathbf{f}}^0.\end{aligned}$$

Therefore,

$$\tilde{\mathbf{F}}(x,y) = \mathbf{B}(x)\mathbf{F}(x,y)\mathbf{B}^{-1}(x).$$

□

1.1.2 The Lax Equation

In the case when $\mathcal{V} = \mathfrak{g}$, we obtain an special type of linear system in which the matrix $\mathbf{U}(x)$ takes values in the adjoint algebra of $\mathfrak{g}$. First, we recall the definition of adjoint group and adjoint algebra.

Let $\mathfrak{g}$ be the Lie algebra of G. The **adjoint representation** of G on $\mathfrak{g}$ is a homomorphism

$$\begin{aligned} \mathrm{Ad} \quad &: \quad G \to \mathbf{GL}(\mathfrak{g}) \\ g \quad &\mapsto \quad \mathrm{Ad}_g \\ \mathrm{Ad}_g(X) \quad &\stackrel{\mathrm{def}}{=} \quad \frac{\mathrm{d}}{\mathrm{d}t} g \exp(tX)\, g^{-1}|_{t=0}. \end{aligned} \tag{1.13}$$

The **adjoint operator** of $\mathfrak{g}$ on $\mathfrak{g}$ is the homomorphism given by the differential of the adjoint representation (1.13).

$$\mathrm{ad} \quad : \quad \mathfrak{g} \to \mathbf{End}(n) \tag{1.14}$$

$$\mathrm{ad} \quad \stackrel{\mathrm{def}}{=} \quad \mathrm{d}\,(\mathrm{Ad})_e. \tag{1.15}$$

In particular, if G is a matrix group, then

$$\mathbf{Ad}_X(Y) := XYX^{-1}, \tag{1.16}$$

and the adjoint operator is the matrix commutator

$$\mathrm{ad}_X(Y) \stackrel{\mathrm{def}}{=} [X, Y] = XY - YX. \tag{1.17}$$

The *adjoint group* $\mathrm{Ad}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is the subgroup of $\mathbf{GL}(\mathfrak{g})$ generated $\mathrm{e}^{(\mathrm{ad}_X)}$, with X run over $\mathfrak{g}$. The corresponding Lie algebra of the adjoint group is said to be *adjoint algebra* and denoted by $\mathrm{ad}(\mathfrak{g})$.

Proposition 1.1.4. *Let $\mathfrak{g}$ be a subalgebra of $\mathbf{gl}(n, \mathbb{F})$. Fix a smooth map $\mathbf{U}(x) \in \mathrm{ad}(\mathfrak{g})$ which is bounded on $\mathbb{R}$. Then the fundamental solution $\mathbf{F}(x, y)$ of the corresponding linear system (1.1) takes values in the adjoint group,*

$$\mathbf{F}(x, y) \in \mathrm{Ad}(\mathfrak{g}) \quad \forall\ x, y. \tag{1.18}$$

Proof. By Proposition 1.1.2, $\mathbf{F}(x, y)$ belongs to the connected component of a Lie associated to the Lie algebra $\mathrm{ad}(\mathfrak{g})$. The connected component is precisally $\mathrm{Ad}(G^0)$, where G^0 is the connected component of G. According to [7], we have the following fact: the adjoint group of g is the unique connected Lie subgroup of $\mathbf{GL}(\mathfrak{g})$ with Lie algebra equal to $\mathrm{ad}(\mathfrak{g})$. Hence,

$$\mathrm{Ad}(G^0) = \mathrm{Ad}(\mathfrak{g}). \tag{1.19}$$

Therefore, (1.19) implies that $\mathbf{F}(x, y)$ is in the adjoint group. □

Under the hypothesis of Proposition 1.1.4, the data $\mathbf{U}$ and $\mathbf{f}$ in (1.1) can be represented as follows

$$\mathbf{U}(x) = \mathrm{Ad}_{\mathbf{A}(x)} \quad \mathbf{f}(x) = \mathbf{L}(x); \tag{1.20}$$

where $\mathbf{A}(x) \in \mathfrak{g}$ and $\mathbf{L}(x) \in \mathrm{gl}(\mathfrak{g})$. Therefore, system (1.1) takes the form

$$\frac{\mathrm{dL}}{\mathrm{d}x} = [\mathbf{L}, \mathbf{A}], \tag{1.21}$$

This system is called the *Lax equation* [11]. In this case, condition (1.2) reads

$$\|\mathbf{A}(x)\| < \infty. \tag{1.22}$$

Proposition 1.1.5. *Under condition (1.22), the solution $\mathbf{L}(x)$ of Lax equation with initial data $\mathbf{L}(0) = \mathbf{L}_0$ is well defined for all $x \in \mathbb{R}$ and given by*

$$\mathbf{L}(x) = \mathrm{Ad}_{\Phi(x)}\mathbf{L}_0 = \Phi(x)\mathbf{L}_0\Phi^{-1}(x). \tag{1.23}$$

Where Φ is the fundamental solution of the linear system associated with (1.21),

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = -\mathbf{A}\Phi, \tag{1.24}$$

$$\Phi(0) = \mathbf{I}. \tag{1.25}$$

Proof. Since $\|\mathbf{A}(x)\| < \infty$, there exists the fundamental solution $\Phi(x)$ in (1.24)-(1.25). Define

$$\widetilde{\mathbf{L}}(x) = \Phi(x)\mathbf{L}_0\Phi^{-1}(x). \tag{1.26}$$

Differentiating (1.26)

$$\frac{\mathrm{d}\widetilde{\mathbf{L}}}{dx} = \frac{\mathrm{d}\Phi}{dx}\mathbf{L}_0\Phi^{-1} - \Phi\mathbf{L}_0\Phi^{-1}\frac{\mathrm{d}\Phi}{\mathrm{d}x}\Phi^{-1} \tag{1.27}$$

$$= \Phi\mathbf{L}_0\Phi^{-1}\mathbf{A} - \mathbf{A}\Phi\mathbf{L}_0\Phi^{-1} \tag{1.28}$$

$$= \left[\widetilde{\mathbf{L}}, \mathbf{A}\right], \tag{1.29}$$

and $\widetilde{\mathbf{L}}(0) = \mathbf{L}_0$. Therefore $\widetilde{\mathbf{L}}$ satisfy the Lax equation (1.21); and hence, by the uniqueness property, we have $\mathbf{L}(x) = \widetilde{\mathbf{L}}(x)$. □

Corollary 1.1.6. *The eigenvalues of $\mathbf{L}(x)$, $\mathrm{tr}\,\mathbf{L}^k(x)$ and $\det \mathbf{L}^k(x)$ do not depend on x, that is, they are first integral of the Lax equation.*

Proof. By Proposition 1.1.5 we have that for all x $\mathbf{L}(x)$ is conjugate to $\mathbf{L}_0$. Hence the eigenvalues of $\mathbf{L}(x)$ are the same to eigenvalues of $\mathbf{L}_0$ [6]. The rest of the corollary follows from basic properties of the trace and determinant:

$$\mathrm{tr}(\mathbf{L}(x)) = \mathrm{tr}(\Phi(x)\mathbf{L}_0\Phi^{-1}(x)) = \mathrm{tr}(\Phi(x)\Phi^{-1}(x)\mathbf{L}_0) = \mathrm{tr}(\mathbf{L}_0), \tag{1.30}$$

and

$$\det(\mathbf{L}(x)) = \det(\Phi(x)\mathbf{L}_0\Phi^{-1}(x)) = \det(\Phi(x)\Phi^{-1}(x)\mathbf{L}_0) = \det(\mathbf{L}_0). \tag{1.31}$$

Thus, they do not depend on x. □

Finally, we observe the following geometric property of the Lax equation. Let us denote by

$$O_V \stackrel{\text{def}}{=} \{\mathrm{Ad}_g(V) | g \in G\} = \mathrm{Ad}(G)V \tag{1.32}$$

the adjoint orbit of $V \in \mathfrak{g}$

Corollary 1.1.7. *If L is a solution of the Lax equation (1.21), then $L(x) \in O_{L_0}$ for all x.*

Follows directly from the Proposition 1.1.5 and the definition of adjoint orbit.

1.1.3 The Zero Curvature Equation

Let $\mathfrak{g}$ be a Lie algebra of a Lie group G and $\mathbf{U}, \mathbf{V} : \mathbb{R}^2_{(x,t)} \to \mathfrak{g}$ smooth functions. The equation

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U}, \mathbf{V}] = 0, \tag{1.33}$$

is called the *zero curvature equation.* This equation arises in many branch of mathematics [8, 11]. In the theory of differential equation it is the compatibility condition of the two linear systems

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x} = \mathbf{U}(x,t)\mathbf{f}, \tag{1.34}$$

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} = \mathbf{V}(x,t)\mathbf{f}, \tag{1.35}$$

$$\mathbf{f}(x_0, t_0) = \mathbf{f}^0. \tag{1.36}$$

The name of zero curvature comes from the connection theory. Let $\alpha = \mathbf{U}\mathrm{d}x + \mathbf{V}\mathrm{d}t$ be an 1-form in $\mathbb{R}^2$ with values in $\mathfrak{g}$. The zero curvature equation (1.33) is equivalent to

$$\mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0, \tag{1.37}$$

which is known as the *Maurer-Cartan equation.* If α is a connection form on the G-bundle $\mathbb{R}^2 \times G$, then the Maurer-Cartan equation says that the curvature of the connection of α is identically zero (see [11]).

Proposition 1.1.8. *Let G be a Lie group. Let $\alpha = \mathbf{U}\mathrm{d}x + \mathbf{V}\mathrm{d}t$ be an 1-form in $\mathbb{R}^2$, where $\mathbf{U}, \mathbf{V} : \mathbb{R}^2_{(x,t)} \to \mathfrak{g}$. The following statements are equivalents:*

1. *$\mathbf{U}$ and $\mathbf{V}$ satisfy the zero curvature equation (1.33).*

2. *There exist a function $\mathcal{F} : \mathbb{R}^2 \to G$ such that $\alpha = \mathcal{F}^{-1}\mathrm{d}\mathcal{F}$.*

3. *For each point $(x_0, t_0) \in \mathbb{R}^2$, there exist an open neighborhood $U \subset \mathbb{R}^2$ of (x_0, t_0) and a function $\mathbf{f} : U \to \mathbb{R}^2$ satisfying the coupled linear problem (1.34)-(1.36).*

The proof of this result is out of the scope of this text, and can be found in [11].

There is a closely relation between Lax equations and zero curvature equation. Let $\mathbf{U}, \mathbf{V} : \mathbb{R}^2_{(x,t)} \to = \mathrm{ad}(\mathfrak{g})$ be smooth functions. Consider the following coupled Lax equations

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}x} = [\mathbf{L}, \mathbf{U}], \tag{1.38}$$

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = [\mathbf{L}, \mathbf{V}]. \tag{1.39}$$

The compatibility condition for the existence of $\mathbf{L}(x,t)$, in (1.38),(1.39), leads to the following fact: $\mathbf{U}$ and $\mathbf{V}$ must satisfy the zero curvature equation (1.33). Moreover, there also exists a relationship between the solution $\mathbf{L}(x,t)$, with $\mathbf{L}(0,0) = \mathbf{L}_0$ and the function $\mathcal{F}$ in Proposition 1.1.8, which is

$$\mathbf{L}(x,t) = \mathrm{Ad}_{\mathcal{F}^{-1}(x,t)}\mathbf{L}_0. \tag{1.40}$$

1.1.4 Lyapunov Transformations

We can introduce an equivalence relation for linear systems using the Lyapunov transformation. The results on the reducibility, stated later in this chapter, are preserved under Lyapunov transformation.

Definition 1.1.1. *The change of coordinates*

$$\widetilde{\mathbf{f}}(x) = \mathbf{T}(x)\mathbf{f}(x), \tag{1.41}$$

is called a Lyapunov transformation if the C^∞ function $x \mapsto \mathbf{T}(x) \in G$ satisfies the following conditions:

(i) $\mathbf{T}(x)$ and $\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}x}$ are bounded,

$$\sup_{x\in\mathbb{R}} \|\mathbf{T}(x)\| < \infty \quad \textit{and} \quad \sup_{x\in\mathbb{R}} \left\| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}x}(x) \right\| < \infty; \tag{1.42}$$

(ii) there exist a real number m such that $|\det \mathbf{T}(x)| \geq m > 0$ for all $x \in \mathbb{R}$.

We remark that the inverse of a Lyapunov transformation is also a Lyapunov transformation. Indeed, we have

$$|\det \mathbf{T}(x)| \le M < \infty. \tag{1.43}$$

It follows that

$$|\det \mathbf{T}^{-1}(x)| = \frac{1}{|\det \mathbf{T}(x)|} \ge \frac{1}{M}. \tag{1.44}$$

Using

$$\mathbf{T}^{-1}(x) = \frac{1}{|\det \mathbf{T}(x)|}\left[\Delta_{ij}\right], \tag{1.45}$$

by (i), (ii) we get that

$$\sup_{x\in\mathbb{R}} \|\mathbf{T}^{-1}(x)\| < \infty \text{ and } \sup_{x\in\mathbb{R}} \left\| \frac{\mathrm{d}\mathbf{T}^{-1}}{\mathrm{d}x}(x) \right\| < \infty. \tag{1.46}$$

Moreover, it is easy to see that the composition of two Lyapunov transformation is also Lyapunov.

Definition 1.1.2. *Two linear systems*

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x} = \mathbf{U}(x)\mathbf{f} \quad \mathbf{U}(x) \in \mathfrak{g},\ \|\mathbf{U}(x)\| < \infty, \tag{1.47}$$

$$\frac{\mathrm{d}\widetilde{\mathbf{f}}}{\mathrm{d}x} = \widetilde{\mathbf{U}}(x)\widetilde{\mathbf{f}} \quad \widetilde{\mathbf{U}}(x) \in \mathfrak{g},\ \|\widetilde{\mathbf{U}}(x)\| < \infty. \tag{1.48}$$

are equivalent (in the sense of Lyapunov) if they exist a Lyapunov transformation $\mathbf{T}(x)$ *such that*

$$\widetilde{\mathbf{f}}(x) = \mathbf{T}(x)\mathbf{f}(x).$$

The Lyapunov transformation gives an equivalence relation in the set of linear systems.

Definition 1.1.3. *A linear system*

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x} = \mathbf{U}(x)\mathbf{f}, \quad \mathbf{U}(x) \in \mathfrak{g} \tag{1.49}$$

is called ***reducible*** *if it is equivalent to a system with constant coefficients*

$$\frac{\mathrm{d}\widetilde{\mathbf{f}}}{\mathrm{d}x} = \mathbf{K}\widetilde{\mathbf{f}}, \quad \mathbf{K} \in \mathfrak{g}. \tag{1.50}$$

Lyapunov's Criterion. The linear system (1.75) is reducible if and only if its fundamental solution $\mathbf{F}(x, y)$ has a representation

$$\mathbf{F}(x, y) = \mathbf{T}(x)\mathrm{e}^{x\mathbf{K}(y)}; \tag{1.51}$$

where $\mathbf{T}$ is a Lyapunov transformation and $\mathbf{K}(y) \in \mathfrak{g}$ does not depend on x.

1.2 The Definition of Monodromy for Linear Boundary Problems

Here we define the *monodromy matrix* for linear systems with some boundary conditions and discuss its main properties. We shall consider the following three conditions: *periodic, quasi-periodic and decreasing* .
Let G be a subgroup of $\mathbf{GL}(n, \mathbb{F})$ and $\mathfrak{g}$ its Lie algebra. We let consider the linear system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}(x,y) &= \mathbf{U}(x)\mathbf{F}(x,y), & (1.52)\\ \mathbf{F}(x,y)|_{x=y} &= \mathbf{I}. & (1.53) \end{aligned}$$

1.2.1 The Periodic Case

The linear system (1.52) is periodic if there exists a real number $L > 0$ such that the matrix coefficients satisfies $\mathbf{U}(x + 2L) = \mathbf{U}(x)$ for each $x \in \mathbb{R}$. We will define the monodromy matrix in relation with the fundamental solution of the system (1.52). Although the material exposed here is well known [15, 17] , this case works as a model to define the monodromy matrix in the quasi-periodic and decreasing cases. The main result for periodic coefficient is the **Floquet theorem**, which states that by means of a change of variables one can reduce equation (1.52) to a system with constant coefficients.

Definition 1.2.1. *Let $\mathbf{F}(x, y)$ be the fundamental solution of the periodic linear problem (1.52). The matrix*

$$\mathbf{M}(y) \stackrel{\text{def}}{=} \mathbf{F}(y + 2L, y) \qquad (1.54)$$

*is called the **monodromy matrix**.*

By Proposition 1.1.1, we have that $\mathbf{F}(x, y) \in G^0$ for each $x \in \mathbb{R}$ and hence $\mathbf{M}(y) \in G^0$.

Proposition 1.2.1. *Let $\mathbf{F}(x, y)$ be the fundamental solution of the periodic linear problem (1.52).*

1. *The function $\mathbf{F}(x + 2L, y)$ is solution of the equation (1.52).*

2. *The fundamental solution $\mathbf{F}(x, y)$ of (1.52) and the map $\mathbf{F}(x+2L, y)$ are related by the equation*

$$\mathbf{F}(x + 2L, y) = \mathbf{F}(x, y)\mathbf{M}(y), \qquad (1.55)$$

where $\mathbf{M}(y)$ is the monodromy matrix.

Proof. 1. Since $\mathbf{F}(x,y)$ is fundamental solutions, by (1.52) we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}(x+2L,y) = \mathbf{U}(x+2L)\mathbf{F}(x+2L,y).$$

So, $\mathbf{U}(x)$ is $2L$-periodic and

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}(x+2L,y) = \mathbf{U}(x)\mathbf{F}(x+2L,y).$$

Hence $\mathbf{F}(x+2L,y)$ is also a solution of (1.52).

2. By (1) we have the function $\mathbf{F}(x,y)$ and $\mathbf{F}(x+2L,y)$ are both solution of the same differential equation. From basic facts about theory of ODE [15],[17], it follows that $\mathbf{F}(x,y)$ and $\mathbf{F}(x+2L,y)$ are linearly dependent. This implies the existence of a constant matrix $\mathbf{C}$, such that

$$\mathbf{F}(x+2L,y) = \mathbf{F}(x,y)\mathbf{C}. \tag{1.56}$$

Putting $x = y$, we obtain

$$\mathbf{C} = \mathbf{F}(2L+y,y) = \mathbf{M}(y)$$

□

Now, we recall the Floquet theorem. According to [7], we have the following fact.

Proposition 1.2.2. *Let $G \subset \mathbf{GL}(n,\mathbb{C})$ be a subgroup and $\mathbf{g} \subset \mathbf{gl}(n,\mathbb{C})$ its Lie algebra. Then*

$$\exp(\mathfrak{g}) = G^0, \tag{1.57}$$

where G^0 is connected component of G.

Since $\mathbf{M}(y) \in G^0$, from Proposition (1.2.2), there exist a matrix $\mathbf{K} \in \mathfrak{g}$ such that

$$\mathbf{K} = \frac{1}{2L}\ln \mathbf{M}. \tag{1.58}$$

Theorem 1.2.3 (Floquet-Lyapunov). *Let $\mathbf{F}(x,y)$ be fundamental solution of the linear problem (1.52),(1.53), with periodic coefficients in $\mathfrak{g} \subset \mathbf{gl}(n,\mathbb{C})$. Then, the fundamental solution can be expressed in the form*

$$\mathbf{F}(x,y) = \Psi(x,y)\mathrm{e}^{x\mathbf{K}(\mathbf{y})}, \tag{1.59}$$

where $\mathbf{K}$ is given by (1.58) and Ψ is a matrix function with the following properties:

1. $\Psi(x+2L,y) = \Psi(x,y)$,

2. $\Psi(x, y) \in G^0$,

for all x.

Proof. Let us define

$$\Psi(x, y) \stackrel{\text{def}}{=} \mathbf{F}(x, y)\mathrm{e}^{-x\mathbf{K}(y)}. \tag{1.60}$$

It is clear that $\Psi(x, y)$ is in G^0 because is the product of $\mathbf{F}(x, y)$ with $\mathrm{e}^{-x\mathbf{K}(y)}$ both belong to G^0. We have to check that Ψ is periodic,

$$\begin{aligned}\Psi(x + 2L, y) &= \mathbf{F}(x + 2L, y)\mathrm{e}^{-(x+2L)\mathbf{K}(y)} \\ &= \mathbf{F}(x, y)\mathbf{M}(y)\mathrm{e}^{-2L\mathbf{K}(y)}\mathrm{e}^{-x\mathbf{K}(y)} \\ &= \mathbf{F}(x, y)\mathrm{e}^{-x\mathbf{K}(y)} \\ &= \Psi(x, y).\end{aligned}$$

□

The classical Floquet-Lyapunov reducibility theorem provides a periodic change of variable which reduces equation (1.52) to a system with constant coefficient.As a consecuence of Theorem 1.2.3, we get the following analogue of the Floquet-Lyapunov theorem reformulated in terms of the fundamental solution.

Theorem 1.2.4 (Lyapunov reducibility). *Let* $\mathbf{K}(y)$ *given by (1.58) and* $\Psi(x, y)$ *by (1.59). Under the* $2L$*-periodic change of variables*

$$\widetilde{\mathbf{F}}(x, y) = \Psi^{-1}(x, y)\mathbf{F}(x, y)\Psi(y, y) \tag{1.61}$$

the system (1.52) reduces to the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{\mathbf{F}}(x, y) &= \mathbf{K}(y)\widetilde{\mathbf{F}}(x, y), &(1.62)\\ \widetilde{\mathbf{F}}(x, y)|_{x=y} &= \mathbf{I}. &(1.63)\end{aligned}$$

Proof. It follows from (1.61) that

$$\frac{\mathrm{d}\widetilde{\mathbf{F}}}{\mathrm{d}x}(x, y) = -\Psi^{-1}(x, y)\frac{\mathrm{d}\Psi}{\mathrm{d}x}(x, y)\Psi^{-1}(x, y)\mathbf{F}\Psi(y, y) + \Psi^{-1}(x, y)\mathbf{U}(x)\mathbf{F}\Psi(y, y)$$

On other hand, differentiating equation (1.60) gives

$$\frac{\mathrm{d}\Psi}{\mathrm{d}x}(x, y) = \mathbf{U}(x)\Psi(x, y) - \Psi(x, y)\mathbf{K}(y).$$

Substituting the last equation into the other one, we get

$$\begin{aligned}\frac{d\widetilde{\mathbf{F}}}{dx} &= - \Psi^{-1}(x,y)\mathbf{U}(x)\Psi(x,y)\widetilde{\mathbf{F}} + \mathbf{K}(y)\widetilde{\mathbf{F}} + \Psi^{-1}(x,y)\mathbf{U}(x)\Psi(x,y)\widetilde{\mathbf{F}} \\ \frac{d\widetilde{\mathbf{F}}}{dx} &= \mathbf{K}(y)\widetilde{\mathbf{F}}.\end{aligned}$$

Now, putting $x = y$ into (1.61) leads to

$$\widetilde{\mathbf{F}}(y,y) = \Psi^{-1}(y,y)\mathbf{F}(y,y)\Psi(y,y) = \Psi^{-1}(y,y)\Psi(y,y) = \mathbf{I}. \tag{1.64}$$

□

As it was established before, if $\mathbf{U}(x) \in \mathbf{gl}(n,\mathbb{R})$ then $\mathbf{F}(x,y)$ belongs to $\mathbf{GL}(n,\mathbb{R})$, but in general the matrices $\Psi(x,y)$ and $\mathbf{K}$ in (1.59) take values in the complex Lie group and complex Lie algebra, respectively. In this case, the Floquet-Lyapunov Theorem 1.2.3 is still true under the following assumption.

Corollary 1.2.5. *If the matrix coefficient in Theorem (1.2.3) takes values in* $\mathfrak{g} \subset \mathbf{gl}(n,\mathbb{R})$ *then fundamental solution has the form (1.59) only if*

$$\mathbf{M}(y) \in \exp(\mathfrak{g}) \tag{1.65}$$

Proof. The condition (1.65) guarantees that $\mathbf{K} \in \mathbf{gl}(n,\mathbb{R})$, this proves the statement.

□

1.2.2 The Quasi-periodic Case

The linear system (1.52) is called **quasi-periodic** if there exist a real number $L > 0$ and a constant matrix $\mathbf{Q} \in G^0$ such that the matrix coefficient $\mathbf{U}$ satisifes the condition

$$\mathbf{U}(x+2L) = \mathbf{Q}^{-1}\mathbf{U}(x)\mathbf{Q} \tag{1.66}$$

for all $x \in \mathbb{R}$.

Definition 1.2.2. *Let* $\mathbf{F}(x,y)$ *be the fundamental solution of the quasi-periodic linear problem (1.52). The matrix*

$$\mathbf{M}(y) \stackrel{\text{def}}{=} \mathbf{F}(y+2L,y) \tag{1.67}$$

is called the ***monodromy****.*

In the quasi-periodic case, the monodromy $\mathbf{M}(x)$ has the properties.

Proposition 1.2.6. *Let* $\mathbf{F}(x,y)$ *be the fundamental solution of the quasi-periodic linear problem (1.52) and* $\mathbf{M}(y)$ *its monodromy matrix. We have the following relations:*

(a)

$$\mathbf{F}(x+2L,y) = \mathbf{Q}^{-1}\mathbf{F}(x,y)\mathbf{Q}\mathbf{M}(y), \tag{1.68}$$

(b)

$$\mathbf{F}(x+2L,y+2L) = \mathbf{Q}^{-1}\mathbf{F}(x,y)\mathbf{Q}, \tag{1.69}$$

(c)

$$\mathbf{Q}\mathbf{M}(x) = \mathbf{F}^{-1}(0,x)\mathbf{Q}\mathbf{M}(0)\mathbf{F}(0,x). \tag{1.70}$$

(d) The monodromy matrix satisfies the quasi-periodicity

$$\mathbf{M}(y+2L) = \mathbf{Q}^{-1}\mathbf{M}(y)\mathbf{Q} \quad \forall y \in \mathbb{R}. \tag{1.71}$$

Proof. (a) The matrix function $\mathbf{G}(x,y) = \mathbf{F}(x,y)\mathbf{Q}\mathbf{M}(y)$ is a solution of (1.3), because $\mathbf{F}(x,y)$ is its fundamental solution, whenever y is fixed. Evaluating at $x = y$, we have $\mathbf{G}(y,y) = \mathbf{Q}\mathbf{M}(y)$. On other hand, as we have established in Proposition (1.2.1), the function $\mathbf{H}(x,y) = \mathbf{Q}\mathbf{F}(x+2L,y)$ is also a fundamental solution, which satisfies the same boundary condition, $\mathbf{H}(y,y) = \mathbf{Q}\mathbf{M}(y) = \mathbf{G}(y,y)$. Then, $\mathbf{H}(x,y) = \mathbf{G}(x,y)$.

(b) Using the transition property, by Proposition (1.1.1), we have

$$\begin{aligned}\mathbf{F}(x+2L,y+2L) &= \mathbf{F}(x+2y,y)\mathbf{F}(y,y+2L),\\ &= \mathbf{F}(x+2L,y)\mathbf{M}^{-1}(y).\end{aligned}$$

It follows from (a) that

$$\mathbf{F}(x+2L,y+2L) = \mathbf{Q}^{-1}\mathbf{F}(x,y)\mathbf{Q}\mathbf{M}(y)\mathbf{M}^{-1}(y)$$

(c) By the definition, $\mathbf{M}(0) = \mathbf{F}(2L,0)$. Now, we have to prove the identity

$$\mathbf{F}(x,0)\mathbf{Q}\mathbf{F}(2L,0)\mathbf{F}(0,x) = \mathbf{F}(x,0)\mathbf{Q}\mathbf{F}(2L,x).$$

By (b), we obtain

$$\begin{aligned}\mathbf{F}^{-1}(0,x)\mathbf{Q}\mathbf{M}(0)\mathbf{F}(\mathbf{0},\mathbf{x}) &= \mathbf{Q}\mathbf{F}(x+2L,2L)\mathbf{Q}^{-1}\mathbf{Q}\mathbf{F}(2L,x)\\ &= \mathbf{Q}\mathbf{F}(x+2L,x)\\ &= \mathbf{Q}\mathbf{M}(x).\end{aligned}$$

(d) Directly from definition of monodromy (1.67) we have

$$\mathbf{M}(y+2L) = \mathbf{F}(y+4L,y+2L). \tag{1.72}$$

By part (b)

$$\begin{aligned}\mathbf{M}(y+2L) &= \mathbf{Q}^{-1}\mathbf{F}(y+2L,y)\mathbf{Q} &(1.73)\\ &= \mathbf{Q}^{-1}\mathbf{M}(y)\mathbf{Q} &(1.74)\end{aligned}$$

□

Now, we formulate an analogue of the Floquet-Lyapunov theorem 1.2.3. First, we discuss some facts about one class of transformations which reduce a quasi-periodic linear problem (1.66) into a periodic one. Consider the linear system (1.1)

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x} = \mathbf{U}(x)\mathbf{f}, \ \mathbf{f} = \mathbf{f}(x,y) \in \mathbb{F}^n \tag{1.75}$$

$$\mathbf{f}(y,y) = \mathbf{f}^0, \tag{1.76}$$

where $\mathbf{U}(x)$ is a quasi-periodic matrix. Let $\mathbf{T}(x) \in G^0$ be a smooth function such that

$$\mathbf{T}(x+2L) = \mathbf{T}(x)\mathbf{Q}. \tag{1.77}$$

and consider the change of variables

$$\widetilde{\mathbf{f}}(x,y) = \mathbf{T}(x)\mathbf{f}(x,y). \tag{1.78}$$

Lemma 1.2.7. *Let* $\mathbf{f}(x,y)$ *be the solution of the quasi-periodic system (1.75). Then, the function* $\widetilde{\mathbf{f}}(x,y)$ *in (1.78) is a solution of a periodic system.*

Proof. Differentiating (1.78), we obtain that $\widetilde{\mathbf{f}}(x,y)$ satisfy the system

$$\frac{\mathrm{d}\widetilde{\mathbf{f}}}{\mathrm{d}x} = \widetilde{\mathbf{U}}(x)\widetilde{\mathbf{f}}, \tag{1.79}$$

where

$$\widetilde{\mathbf{U}} = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}x}\mathbf{T}^{-1} + \mathbf{T}\mathbf{U}\mathbf{T}^{-1}.$$

We only must check that $\widetilde{\mathbf{U}}(x)$ is periodic,

$$\begin{aligned}\widetilde{\mathbf{U}}(x+2L) &= \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}x}(x+2L)\mathbf{T}^{-1}(x+2L) + \mathbf{T}(x+2L)\mathbf{U}(x+2L)\mathbf{T}^{-1}(x+2L)\\ &= \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}x}(x)\mathbf{Q}\mathbf{Q}^{-1}\mathbf{T}^{-1}(x) + \mathbf{T}(x)\mathbf{Q}\mathbf{Q}^{-1}\mathbf{U}(x)\mathbf{Q}\mathbf{Q}^{-1}\mathbf{T}^{-1}(x)\\ &= \widetilde{\mathbf{U}}(x).\end{aligned}$$

□

Since both $\mathbf{Q}$ and $\mathbf{M}(y)$ belong to the subgroup $G^0 \subset \mathbf{GL}(n,\mathbb{C})$, by the Proposition 1.2.2 there exists a matrix $\mathbf{K} \in \mathfrak{g}$ such that

$$\mathbf{K}(y) = \frac{1}{2L}\ln \mathbf{QM}(\mathbf{y}). \tag{1.80}$$

Proposition 1.2.8. *Let* $\mathbf{F}(x,y)$ *be fundamental solution of the linear problem (1.52),(1.53), with quasi-periodic coefficient* $\mathbf{U}$ *in* $\mathfrak{g} \subset \mathbf{gl}(n,\mathbb{C})$. *Then, the fundamental solution can be expressed in the form*

$$\mathbf{F}(x,y) = \Psi(x,y)\mathrm{e}^{x\mathbf{K}(\mathbf{y})}, \tag{1.81}$$

where $\mathbf{K}$ *is given by (1.80) and* Ψ *is a matrix function with the following properties:*

$$\Psi(x+2L,y) = \mathbf{Q}^{-1}\Psi(x,y), \tag{1.82}$$

and

$$\Psi(x,y) \in G^0, \tag{1.83}$$

for all x.

Proof. Let

$$\Psi(x,y) \stackrel{\mathrm{def}}{=} \mathbf{F}(x,y)\mathrm{e}^{-x\mathbf{K}(y)}. \tag{1.84}$$

Then, $\Psi(x,y)$ is in G^0. We have to check that Ψ satisfies the property (1.82)

$$\begin{aligned}
\Psi(x+2L,y) &= \mathbf{F}(x+2L,y)\mathrm{e}^{-(x+2L)\mathbf{K}(y)} \\
&= \mathbf{Q}^{-1}\mathbf{F}(x,y)\mathbf{Q}\mathbf{M}(y)\mathrm{e}^{-2L\mathbf{K}(y)}\mathrm{e}^{-x\mathbf{K}(y)} \\
&= \mathbf{Q}^{-1}\mathbf{F}(x,y)\mathrm{e}^{-x\mathbf{K}(y)} \\
&= \mathbf{Q}^{-1}\Psi(x,y).
\end{aligned}$$

□

Let us take $\mathbf{T}(x,y) = \Psi^{-1}(x,y)$, where $\Psi^{-1}(x,y)$ is a matrix function satisfying the properties (1.82),(1.83). $\mathbf{T}(x,y)$ satisfies condition (1.77). Hence, by Lemma 1.2.7 $\mathbf{T}$ gives a change of coordinates that reduces the quasi-periodic linear system (1.52),(1.66) to a system with periodic coefficient. From here, we get the follows results to establish in the next result.

Theorem 1.2.9. *Let* $\mathbf{K}(y)$ *and* $\Psi(x,y)$ *be matrix valued function given by (1.80) and (1.81), respectively. Under the transformation*

$$\widetilde{\mathbf{F}}(x,y) = \Psi^{-1}(x,y)\mathbf{F}(x,y)\Psi(y,y) \tag{1.85}$$

quasi-periodic system (1.52),(1.66) is reduced to the system with constant coefficients

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{\mathbf{F}}(x,y) &= \mathbf{K}(y)\widetilde{\mathbf{F}}(x,y), & (1.86)\\
\widetilde{\mathbf{F}}(x,y)|_{x=y} &= \mathbf{I}. & (1.87)
\end{aligned}$$

The proof is proceeded in the same way as in the periodic case.

Summarizing the above results, we get the follows conclusions:

- In the periodic case, the fundamental solution $\mathbf{F}(x,y)$ is of the form (1.51), where $\Psi(x,y)$ is periodic. In particular, Ψ and all its derivatives are bounded,

$$\sup_{x\in\mathbb{R}} \|\Psi(x,y)\| = \sup_{x\in[0,2L]} \|\Psi(x,y)\| < \infty. \tag{1.88}$$

  Therefore, every periodic linear system is reducible in the sense of Lyapunov.

- In the quasi-periodic case the situation is a little bit different. A quasi-periodic system is reducible, if additional condition holds. By Proposition 1.2.8, $\mathbf{F}(x,y)$ can be expressed in the form (1.51), where

$$\Psi(x+2L,y) = \mathbf{Q}^{-1}\Psi(x,y). \tag{1.89}$$

  Therefore, a quasi-periodic system is reducible only if

$$\|\mathbf{Q}\| = 1. \tag{1.90}$$

1.2.3 The Decreasing case

Here we study the linear system (1.52) in the case when all elements of the matrix coefficients $\mathbf{U}(x)$ are absolutely integrable functions on $\mathbb{R}^1$, i.e., $\mathbf{U}_{ij} \in \mathbf{L}_1(\mathbb{R}^1)$. This condition means that

$$\int_{-\infty}^{\infty} \|\mathbf{U}(x)\|dx < \infty, \tag{1.91}$$

where $\|\cdot\|$ is a matricial norm. The space of $n\times n$ matrix functions satisfying (1.2.13) will be denoted by $\mathbf{L}_1^{n\times n}(-\infty,\infty)$. It is clear that under this assumption $\mathbf{U}_{ij}(x)$ vanish as $|x| \to \infty$.

The fundamental solution $\mathbf{F}(x,y)$ of (1.52),() exists for all $x,y \in \mathbb{R}$ (see, for example [14]). We shall analyze the properties of the fundamental solution $\mathbf{F}(x,y)$ when both x and y run over the whole real axis.

Proposition 1.2.10. *Let $\mathbf{F}(x,y)$ be the fundamental solution of the linear system (1.52) in the decreasing case. There exist the limits*

$$\mathbf{F}_\pm(x) = \lim_{y\to\pm\infty} \mathbf{F}(x,y), \tag{1.92}$$

for each x in $\mathbb{R}^1$. Moreover, these limits have the integral representation

$$\mathbf{F}_-(x) = \mathbf{I} + \int_{-\infty}^{x} \mathbf{U}(z)\mathbf{F}_-(z)dz, \tag{1.93}$$

$$\mathbf{F}_+(x) = \mathbf{I} - \int_{x}^{\infty} \mathbf{F}_+(z)\mathbf{U}(z)dz. \tag{1.94}$$

Proof. It is clear that $\mathbf{F}(x,y)$ satisfies the integral equation

$$\mathbf{F}(x,y) = \mathbf{I} + \int_y^x \mathbf{U}(z)\mathbf{F}(z,y)dz.$$

Then

$$\|\mathbf{F}(x,y)\| \leq 1 + \int_y^x \|\mathbf{U}(z)\|\|\mathbf{F}(z,y)\|dz,$$

assuming, without loss of generality $\|\mathbf{I}\| = 1$. Using the Gronwall's inequality, we have

$$\|\mathbf{F}(x,y)\| \leq \exp\int_y^x \|\mathbf{U}(z)\|dz.$$

Taking the limit as $y \to -\infty$, gives

$$\|\mathbf{F}_-(x)\| \leq \exp\int_{-\infty}^x \|\mathbf{U}(z)\|dz < \infty,$$

since $\mathbf{U}(x)$ lies on $\mathbf{L}_1^{n\times n}(\mathbb{R})$. Therefore $\mathbf{F}_-(x)$ exists for each real x. Now, we prove the integral representation for $\mathbf{F}_-(x)$. First, we have to show that the matrix function $\mathbf{U}(z)\mathbf{F}_-(z)$, as function of z, belongs to $\mathbf{L}_1(-\infty,x)$ for each x fixed. From the integral equation for the fundamental solution, we have

$$\mathbf{U}(x)\mathbf{F}(x,y) = \mathbf{U}(x) + \mathbf{U}(x)\int_y^x \mathbf{U}(z)\mathbf{F}(z,y)dz.$$

Then

$$\|\mathbf{U}(x)\mathbf{F}(x,y)\| \leq \|\mathbf{U}(x)\| + \|\mathbf{U}(x)\|\int_y^x \|\mathbf{U}(z)\mathbf{F}(z,y)\|dz.$$

Integrating this inequality over $y < s < x$, we obtain

$$\int_y^x \|\mathbf{U}(s)\mathbf{F}(s,y)\|ds \leq \int_y^x \|\mathbf{U}(s)\|ds + \int_y^x \|\mathbf{U}(s)\|\int_y^s \|\mathbf{U}(z)\mathbf{F}(z,y)\|dzds.$$

By Gronwall's inequality, it follows that

$$\begin{aligned}\int_y^x \|\mathbf{U}(s)\mathbf{F}(s,y)\|ds &\leq \int_y^x \|\mathbf{U}(s)\|\mathrm{ds}\left(\exp\left\{\int_y^x \|\mathbf{U}(s)\|ds\right\}\right)\\ &\leq \int_{-\infty}^x \|\mathbf{U}(s)\|\mathrm{ds}\left(\exp\left\{\int_{-\infty}^x \|\mathbf{U}(s)\|ds\right\}\right) < \infty.\end{aligned}$$

From here, we conclude that

$$\int_{-\infty}^x \|\mathbf{U}(z)\mathbf{F}_-(z)\|\mathrm{dz} < \infty.$$

This implies that $\mathbf{U}(z)\mathbf{F}_-(z)$ lies in $\mathbf{L}_1(-\infty, x)$. Taking the limit as $y \to -\infty$ in the integral equation for $\mathbf{F}(x, y)$, we obtain

$$\mathbf{F}_-(x) = \mathbf{I} + \int_{-\infty}^{x} \mathbf{U}(z)\mathbf{F}_-(z)dz.$$

The existence of the limit (1.125) as $y \to \infty$ is established by similar arguments. Using $\mathbf{F}^{-1}(x, y) = \mathbf{F}(y, x)$, we can see that $F_+^{-1}(x)$ exists and has the integral representation

$$\mathbf{F}_+^{-1}(x) = \mathbf{I} + \int_{x}^{\infty} \mathbf{U}(z)\mathbf{F}_+^{-1}(z)dz.$$

By other hand

$$\begin{aligned}\frac{d\mathbf{F}_+(x)}{dx} &= \mathbf{F}_+(x)\frac{d\mathbf{F}_+^{-1}(x)}{dx}\mathbf{F}_+(x)\\ &= \mathbf{F}_+(x)\mathbf{U}(x).\end{aligned}$$

The last expression leads to

$$\mathbf{F}_+(x) = \mathbf{I} - \int_{x}^{\infty} \mathbf{F}_+(z)\mathbf{U}(z)dz.$$

□

Corollary 1.2.11. *The matrix functions* $\mathbf{F}_\pm(x)$ *satisfy the linear problems*

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}_\pm(x) = \mathbf{U}(x)\mathbf{F}_\pm(x) \tag{1.95}$$

with asymptotic conditions

$$\mathbf{F}_\pm(x) \to \mathbf{I} \quad \textit{if} \quad x \to \pm\infty$$

Proof. Differentiating the integral equation for $\mathbf{F}_\pm(x)$ with respect x and taking limits as $y \to \pm\infty$, give us differential equation (1.95) and the asymptotic conditions. □

Now, we are ready to introduce the notion of monodromy in the decreasing case.

Definition 1.2.3. *The matrix*

$$\mathbf{M} = \mathbf{F}_+^{-1}(x)\mathbf{F}_-(x), \tag{1.96}$$

is said to be the monodromy matrix of system (1.52), (1.2.13).

It is not evident from this definition, that $\mathbf{M}$ is constant for all $-\infty < x < \infty$.

Proposition 1.2.12. *The monodromy matrix* $\mathbf{M}$ *has the properties* *(a)* $\mathbf{M}$ *is independent at* x.

(b) $\mathbf{F}(x,y) = \mathbf{F}_+(x)\mathbf{M}\mathbf{F}_-^{-1}(y)$

Proof. (a) We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{M} &= \frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{F}_+^{-1}(x))\mathbf{F}_-(x) + \mathbf{F}_+^{-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{F}_-(x)) \\ &= -\mathbf{F}_+^{-1}\mathbf{U}(x)\mathbf{F}_-(x) + \mathbf{F}_+^{-1}(x)\mathbf{U}(x)\mathbf{F}_-(x) \\ &= \mathbf{0}.\end{aligned}$$

b) We only need to verify the matrix function $\mathbf{F}_+(x)\mathbf{M}\mathbf{F}_-^{-1}(y)$ also satisfies the linear problem with the same boundary condition,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{F}_+(x)\mathbf{M}\mathbf{F}_-^{-1}(y)) = \mathbf{U}(x)\mathbf{F}_+(x)\mathbf{M}\mathbf{F}_-^{-1}(y) \tag{1.97}$$

putting $x = y$, we get

$$\mathbf{F}_+(y)\mathbf{M}\mathbf{F}_-^{-1}(y) = \mathbf{F}_+(y)\mathbf{F}_+^{-1}(y)\mathbf{F}_-(y)\mathbf{F}_-^{-1}(y) = \mathbf{I}$$

□

It follows from the part (b) of the proposition above, that the monodromy matrix has the following useful representation.

Corollary 1.2.13. *The monodromy matrix in () can be expressed as follows*

$$\mathbf{M} = \lim_{x\to\infty}\lim_{y\to-\infty}\mathbf{F}(x,y) \tag{1.98}$$

Proof. By part (b) of the Proposition 1.2.12 and corollary (1.2.11), it follows that

$$\begin{aligned}\lim_{x\to\infty}\lim_{y\to-\infty}\mathbf{F}(x,y) &= \lim_{x\to\infty}\lim_{y\to-\infty}\mathbf{F}_+(x)\mathbf{M}\mathbf{F}_-^{-1}(y) \\ &= \lim_{x\to\infty}\mathbf{F}_+(x)\left(\lim_{y\to-\infty}\mathbf{M}\mathbf{F}_-^{-1}(y)\right) \\ &= \lim_{x\to\infty}\mathbf{F}_+(x)\mathbf{M} \\ &= \mathbf{M}\end{aligned}$$

□

The formula (1.98) can be simplified if we put $x = L$ an $y = -L$

$$\mathbf{M} = \lim_{L\to\infty}\mathbf{F}(L,-L) \tag{1.99}$$

Therefore, $\mathbf{F}(L,-L)$ can be interpreted as the monodromy matrix for the decreasing case.

1.3 Analytic Properties of Fundamental Solution and Monodromy

First, we recall some facts on the linear systems with analytic coefficients. Let $G \subset \mathbf{GL}(n, \mathbb{C})$ be a Lie subgroup and $\mathfrak{g}$ its Lie algebra. Consider the linear system

$$\frac{\mathrm{df}}{\mathrm{d}x} = \left[\mathbf{U}_0(x) + \lambda \mathbf{U}_1(x) + \lambda^2 \mathbf{U}_2(x) + \ldots\right] \mathbf{f}, \qquad (1.100)$$

where $\mathbf{U}_i$ are smooth functions on a compact domain $D \subset \mathbb{R}$ and λ is a complex parameter. We assume that

$$\sum_{k=0}^{\infty} \lambda^k \int_D |\mathbf{U}_k(x)| dx < \infty, \qquad (1.101)$$

for all $|\lambda| < \lambda_0$, that is, the series (1.101) is convergent.

Proposition 1.3.1. *Let $\mathbf{F}(x, y, \lambda)$ be the fundamental solution of (1.100). Under the assumption (1.101), we have that the sequence of functions*

$$\mathbf{F}_0(x, y, \lambda) = \mathbf{I}, \qquad (1.102)$$

$$\mathbf{F}_k(x, y, \lambda) = \int_y^x \left[\sum_{l=0}^{\infty} \lambda^l \mathbf{U}_l(\tau)\right] \mathbf{F}_{k-1}(\tau, \lambda) \mathrm{d}\tau, \quad (l = 1, 2, \ldots) \qquad (1.103)$$

are analytic function and converge uniformly in x, λ to $\mathbf{F}(x, y, \lambda)$. Moreover, if the the matrix coefficients $\mathbf{U}_0(x) + \lambda \mathbf{U}_1(x) + \lambda^2 \mathbf{U}_2(x) + \ldots$ is an entire function on λ, then $\mathbf{F}(x, \lambda)$ is an entire function of λ for a fixed x.

We do not present the proof here, but it can be found in [17].

1.3.1 The Case $\mathbf{U} = \mathbf{U}_0 + \lambda \mathbf{U}_1$

Let consider the case when the matrix coefficient of system (1.100) takes the form

$$\mathbf{U}(x, \lambda) = \mathbf{U}_0(x) + \lambda \mathbf{U}_1, \qquad (1.104)$$

where $\mathbf{U}_0(x) \in \mathfrak{g}$ is a smooth function and $\mathbf{U}_1 \in \mathfrak{g}$ is an invertible constant matrix. Observe that linear system (1.100) with matrix coefficients of the form (1.104) is equivalent to the spectral problem

$$\mathscr{L}\mathbf{f} = \lambda \mathbf{f}, \qquad (1.105)$$

with the differential operator $\mathscr{L} : \mathbf{L}_1(\mathbb{R}^1) \to \mathbf{L}_1(\mathbb{R}^1)$ given by

$$\mathscr{L} = \mathbf{U}_1^{-1} \frac{\mathrm{d}}{\mathrm{d}x} - \mathbf{U}_1^{-1} \mathbf{U}_0(x). \qquad (1.106)$$

Integral Equation

Here, we derive integral equation for the fundamental solution of (1.104). First, we note that linear system (1.104),(1.100) is equivalent to

$$\mathbf{F}(x,y,\lambda)=\mathbf{I}+\int_y^x \mathbf{U}(z,y,\lambda)\mathbf{F}(z,y,\lambda)\mathrm{d}z. \tag{1.107}$$

Let $\mathbf{E}(\mathbf{x},\mathbf{y},\lambda)$ be the fundamental solution of the linear system

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{E}=\lambda\mathbf{U}_1\mathbf{E}, \tag{1.108}$$
$$\mathbf{E}(y,y,\lambda)=\mathbf{I},\quad \text{for each } \lambda\in\mathbb{R}. \tag{1.109}$$

Let $\mathbf{G}(x,y,\lambda)$ and $\mathbf{H}(x,y,\lambda)$ be the fundamental solutions of the following differential equations, respectively

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{G}(x,y,\lambda)=\mathbf{E}(y,x,\lambda)\mathbf{U}_0(x)\mathbf{E}(x,y,\lambda)\mathbf{G}(x,y,\lambda), \tag{1.110}$$
$$\mathbf{G}(x,y,\lambda)|_{x=y}=\mathbf{I},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathbf{H}(x,y,\lambda)=-\mathbf{H}(x,y,\lambda)\mathbf{E}(x,y,\lambda)\mathbf{U}_0(y)\mathbf{E}(y,x,\lambda), \tag{1.111}$$
$$\mathbf{H}(x,y,\lambda)|_{x=y}=\mathbf{I}.$$

Lemma 1.3.2. *The fundamental solution $\mathbf{F}(x,y,\lambda)$ of the linear problem (1.104), (1.100) can be expressed as*

$$\mathrm{F}(x,y,\lambda)=\mathbf{E}(x,y,\lambda)\mathbf{G}(x,y,\lambda) \tag{1.112}$$

and

$$\mathrm{F}(x,y,\lambda)=\mathbf{H}(x,y,\lambda)\mathbf{E}(x,y,\lambda) \tag{1.113}$$

Proof. This proof is straightforward computation. We just have to check that the right hand sides of equations (1.112),(1.113) satisfy the Cauchy problem for $\mathbf{F}(x,y,\lambda)$. Indeed,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{E}(x,y,\lambda)\mathbf{G}(x,y,\lambda) &= \lambda\mathbf{U}_1\mathbf{E}(x,y,\lambda)\mathbf{G}(x,y,\lambda)+\mathbf{U}_0(x)\mathbf{E}(x,y,\lambda)\mathbf{G}(x,y,\lambda)\\ &= \mathbf{U}(x,\lambda)\mathbf{E}(x,y,\lambda)\mathbf{G}(x,y,\lambda).\end{aligned}$$

On other hand,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}\mathbf{H}(x,y,\lambda)\mathbf{E}(x,y,\lambda) &= -\mathbf{H}(x,y,\lambda)\mathbf{E}(x,y,\lambda)\mathbf{U}_0(y)-\mathbf{H}(x,y,\lambda)\mathbf{E}(x,y,\lambda)\lambda\mathbf{U}_1\\ &= -\mathbf{H}(x,y,\lambda)\mathbf{E}(x,y,\lambda)\mathbf{U}(y,\lambda),\end{aligned}$$

This equation implies

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{H}(x,y,\lambda)\mathbf{E}(x,y,\lambda) = \mathbf{U}(x,\lambda)\mathbf{H}(x,y,\lambda)\mathbf{E}(x,y,\lambda). \tag{1.114}$$

Therefore, (1.112),(1.113) hold. □

The following integral representation for $\mathbf{F}(x,y,\lambda)$ will be useful in the our futher analysis.

Proposition 1.3.3. *The fundamental solution* $\mathbf{F}(x,y,\lambda)$ *of de linear problem (1.120)-(1.121) has the following integral representations*

$$\mathbf{F}(x,y,\lambda) = \mathbf{E}(x,y,\lambda) + \int_y^x \mathbf{F}(x,z,\lambda)\mathbf{U}_0(z)\mathbf{E}(z,y,\lambda)\mathrm{d}z, \tag{1.115}$$

$$\mathbf{F}(x,y,\lambda) = \mathbf{E}(x,y,\lambda) + \int_y^x \mathbf{E}(x,z,\lambda)\mathbf{U}_0(z)\mathbf{F}(z,y,\lambda)\mathrm{d}z. \tag{1.116}$$

Proof. First, we note the linear systems (1.110) and (1.111) are equivalent to the integral equations

$$\mathbf{G}(x,y,\lambda) = \mathbf{I} + \int_y^x \mathbf{E}(y,z,\lambda)\mathbf{U}_0(z)\mathbf{E}(z,y,\lambda)\mathbf{G}(z,y,\lambda)\mathrm{d}z, \tag{1.117}$$

$$\mathbf{H}(x,y,\lambda) = \mathbf{I} + \int_y^x \mathbf{H}(x,z,\lambda)\mathbf{E}(x,z,\lambda)\mathbf{U}_0(z)\mathbf{E}(z,x,\lambda)\mathrm{d}z. \tag{1.118}$$

We only must multiply $\mathbf{E}(x,y,\lambda)$ on the left with (1.117) and on the right with (1.118), respectively, so we use the identities (1.112),(1.113). □

The Proposition 1.3.1 says that the fundamental solution $\mathbf{F}(x,y,\lambda)$ is analytic in λ for each x. Using integral representations (1.115),(1.116) and the formula of integration by parts, we derive the asymptotic expansion for $\mathbf{F}(x,y,\lambda)$:

$$\begin{aligned}\mathbf{F}(x,y,\lambda) &= \mathbf{E}(x,y,\lambda) + \sum_{n=1}^{\infty}\frac{\mathbf{F}_n(x,y,\lambda)}{\lambda^n}\mathbf{E}(x,y,\lambda)\\ &\quad + \sum_{n=1}^{\infty}\frac{\widetilde{\mathbf{F}}_n(x,y,\lambda)}{\lambda^n}\mathbf{E}(y,x,\lambda) + O(|\lambda|^{-\infty}).\end{aligned} \tag{1.119}$$

The results presented in this section was derived under the assumption that the linear problem has a compact domain of definition for the variable x. In particular, they are valid in the periodic and quasi-periodic cases. Finally,we conclude the monodromy matrix satisfies the same analytic properties of the fundamental solution in the periodic and quasi-periodic case.

The Decreasing Case

In order to develop the analytic properties of the fundamental solution and the monodromy of linear systems in the decreasing case, we need to make special considerations. First of all, we can not derive the analytic properties of the monodromy from the fundamental solutions like it was possible in the periodic and quasi-periodic case.

Alternatively, we shall restrict our study to real values of λ, and we shall obtain some analytic properties of the monodromy. In some cases, these conditions are sufficient to extend the domain of analyticity of the monodromy. One of them will be discussed in the next chapter.

We consider the linear system

$$\begin{aligned} \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}x} &= \mathbf{U}(x,\lambda), && (1.120)\\ \mathbf{F}(x,y,\lambda)|_{x=y} &= \mathbf{I}, && (1.121) \end{aligned}$$

with matrix valued function $\mathbf{U}(x,\lambda)$ of the form

$$\mathbf{U}(x,\lambda) = \mathbf{U}_0(x) + \lambda\mathbf{U}_1, \tag{1.122}$$

where λ is a real parameter, $\mathbf{U}_0(x) \in L_1^{n\times n}(-\infty,\infty)$ and $\mathbf{U}_1$ is a constant $\mathfrak{g}$-valued matrix.

We need the following fact.

Lemma 1.3.4. *Let* $\mathbf{U}_0(x)$ *be a function in* $\mathbf{L}_1^{n\times n}(-\infty,\infty)$. *Then, the map* $\mathbf{E}(y,x,\lambda)\mathbf{U}_0(x)\mathbf{E}(x,y,\lambda)$ *also belongs to* $L_1^{n\times n}(-\infty,\infty)$.

Proof. We recall that $\mathbf{E}(x,y,\lambda)$ is the fundamental solution of linear system (1.108),(1.109), so that it has the integral representation

$$\mathbf{E}(x,y,\lambda) = \mathbf{I} + \int_y^x \lambda\mathbf{U}_1\mathbf{E}(s,y,\lambda)ds. \tag{1.123}$$

By Gronwall's inequality [15], we have

$$||\mathbf{E}(x,y,\lambda)|| \le \exp\{\lambda\int_y^x ||\mathbf{U}_1||ds\}. \tag{1.124}$$

Thus,

$$\begin{aligned} ||\mathbf{E}(y,x,\lambda)\mathbf{U}_0(x)\mathbf{E}(x,y,\lambda)|| &\le \exp\{\lambda\int_x^y ||\mathbf{U}_1||ds\}||\mathbf{U}_0(x)||\exp\{\lambda\int_y^x ||\mathbf{U}_1||ds\}\\ &\le \exp\{-\lambda\int_y^x ||\mathbf{U}_1||ds\}\exp\{\lambda\int_y^x ||\mathbf{U}_1||ds\}||\mathbf{U}_0(x)||\\ &\le ||\mathbf{U}_0(x)||. \end{aligned}$$

Hence, the function $\mathbf{E}(y,x,\lambda)\mathbf{U}_0(x)\mathbf{E}(x,y,\lambda)$ is absolutely integrable on $\mathbb{R}$. □

Using the result above, we deduce integral equations (1.115),(1.116) in the decreasing case.

Proposition 1.3.5. *Let* $\mathbf{F}(x, y, \lambda)$ *be the fundamental solution of system (1.120),(1.121). Then,*

(i) the matrix functions

$$\mathbf{F}_\pm(x, \lambda) = \lim_{y \to \pm\infty} \mathbf{F}(x, y, \lambda)\mathbf{E}(y, \lambda), \tag{1.125}$$

are well defined for each x *and* λ *in* $\mathbb{R}$.

(ii) $\mathbf{F}_\pm(x, \lambda)$ *satisfy the differential equations*

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}_\pm = \mathbf{U}(x, \lambda)\mathbf{F}_\pm, \tag{1.126}$$

with the asymptotic conditions

$$\mathbf{F}_\pm(x, \lambda) \to \mathbf{E}(x, \lambda) \quad as \quad x \to \pm\infty. \tag{1.127}$$

Proof. (i) By the Lemma 1.3.2, the fundamental solution $\mathbf{F}(x, y, \lambda)$ can be written as

$$\mathbf{F}(x, y, \lambda) = \mathbf{H}(x, y, \lambda)\mathbf{E}(x - y, \lambda).$$

Thus, we have

$$\mathbf{F}(x, y, \lambda)\mathbf{E}(y, \lambda) = \mathbf{H}(x, y, \lambda)\mathbf{E}(x, \lambda).$$

Since the function $\mathbf{H}(x, y, \lambda)$ is the fundamental solution of a linear system (1.111) in the decreasing case , it follows from Proposition 1.2.10 that the limits

$$\mathbf{H}_\pm(x, \lambda) = \lim_{y \to \pm\infty} \mathbf{H}(x, y, \lambda)$$

exist. Therefore,

$$\mathbf{F}_\pm(x, \lambda) = \lim_{y \to \pm\infty} \mathbf{H}(x, y, \lambda)\mathbf{E}(x, \lambda) = \mathbf{H}_\pm(x, \lambda)\mathbf{E}(x, \lambda)$$

is well-defined.

(ii) It follows from integral representations (1.115),(1.116).

□

Definition 1.3.1. *The monodromy matrix of the linear system (1.52),(1.2.13) with spectral parameter* λ *is defined by*

$$\mathbf{M}(\lambda) = \mathbf{F}_+^{-1}(x, \lambda)\mathbf{F}_-(x, \lambda) \tag{1.128}$$

Proposition 1.3.6. *The monodromy matrix* $\mathbf{M}$ *has the properties:*

(a) $\mathbf{M}(\lambda)$ *is independent of* x.

(b) $\mathbf{F}(x, y, \lambda) = \mathbf{F}_+(x, \lambda)\mathbf{M}(\lambda)\mathbf{F}_-^{-1}(y, \lambda)$

Since the properties (a) and (b) are independent of the parameter λ, the proof is similar to the Proposition 1.2.12.

1.4 The Time Evolution of the Monodromy Matrix

Let $\mathfrak{g} \subset \mathbf{gl}(n, \mathbb{C})$ be a Lie subalgebra. Assume that the matrix functions

$$(x, t) \mapsto \mathbf{U}(x, t) \in \mathfrak{g},$$
$$(x, t) \mapsto \mathbf{V}(x, t) \in \mathfrak{g},$$

satisfy the *zero curvature equation*

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U}, \mathbf{V}] = 0. \tag{1.129}$$

As we established above, the zero curvature equation is the compatibility condition of the coupled linear systems

$$\frac{d\mathbf{f}}{dx} = \mathbf{U}(x, t)\mathbf{f}, \tag{1.130}$$
$$\frac{d\mathbf{f}}{dt} = \mathbf{V}(x, t)\mathbf{f}. \tag{1.131}$$

Fix $t \in \mathbb{R}$. Let $\mathbf{F}(x, y, t)$ be the fundamental solution of the linear system

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{F}(x, y, t) = \mathbf{U}(x, t)\mathbf{F}(x, y, t), \tag{1.132}$$
$$\mathbf{F}(x, y, t)|_{x=y} = \mathbf{I}. \tag{1.133}$$

With the boundary conditions presented in the section (1.2), we can define the monodromy matrix $\mathbf{M}(t)$. Now, we discuss the dependence of the monodromy matrix the time variable t. The following proposition describes the time evolution of the fundamental solution.

Proposition 1.4.1. *If* $\mathbf{F}(x, y, t)$ *is the solution of system (1.132),(1.133), then*

$$\frac{\partial}{\partial t}\mathbf{F}(x, y, t) = \mathbf{V}(x, t)\mathbf{F}(x, y, t) - \mathbf{F}(x, y, t)\mathbf{V}(y, t) \tag{1.134}$$

Proof. Differentiating equation (1.132) with respect to t and using the zero curvature equation, we get

$$\begin{aligned}\frac{\partial^2 \mathbf{F}}{\partial x \partial t} &= \frac{\partial \mathbf{U}}{\partial t}\mathbf{F} + \mathbf{U}\frac{\partial \mathbf{F}}{\partial t}, \\ &= \frac{\partial \mathbf{V}}{\partial x}\mathbf{F} + \mathbf{VUF} - \mathbf{UVF} + \mathbf{U}\frac{\partial \mathbf{F}}{\partial t}.\end{aligned}$$

Next, we transform the equation in

$$\frac{\partial^2 \mathbf{F}}{\partial x \partial t} = \frac{\partial}{\partial x}(\mathbf{VF}) + \mathbf{U}(\frac{\partial \mathbf{F}}{\partial t} - \mathbf{VF}).$$

After the substitution of (1.130),(1.131) into this equation, we obtain

$$\frac{\partial}{\partial x}(\frac{\partial \mathbf{F}}{\partial t} - \mathbf{VF}) = \mathbf{U}(\frac{\partial \mathbf{F}}{\partial t} - \mathbf{VF}).$$

Then, we notice that $\frac{\partial \mathbf{F}}{\partial t} - \mathbf{VF}$ is solution of equation (1.132). Therefore, this implies the existence of a non-singular matrix $\mathbf{C}$, independent of x, such that

$$\frac{\partial \mathbf{F}}{\partial t} - \mathbf{VF} = \mathbf{FC},$$

Finally, putting $x = y$, we obtain

$$\mathbf{C}(y,t) = -\mathbf{V}(y,t).$$

□

Therefore, the time evolution of the monodromy matrices in the quasi-periodic and decreasing cases are governed by equations of type Lax (1.21).

Proposition 1.4.2. *Assume that the matrix maps $\mathbf{U}(x,t)$ and $\mathbf{V}(x,t)$ satisfy zero curvature equation (1.129). The following assertions hold:*

(i) if $\mathbf{U}(x,t)$ and $\mathbf{V}(x,t)$ are quasi-periodic,

$$\begin{aligned}\mathbf{U}(x+2L,t) &= \mathbf{Q}^{-1}\mathbf{U}(x,t)\mathbf{Q}, \\ \mathbf{V}(x+2L,t) &= \mathbf{Q}^{-1}\mathbf{V}(x,t)\mathbf{Q},\end{aligned}$$

then the monodromy matrix $\mathbf{M}(y,t)$ in (1.2.2) satisfies the Lax equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{MQ}) = [\mathbf{V}(L,t), \mathbf{MQ}]. \tag{1.135}$$

(ii) If for every fixed t, the matrix function $\mathbf{U}(x,t)$ lies in $\mathbf{L}_1^{n\times n}(-\infty,\infty)$ and

$$\lim_{x\to\pm\infty} \mathbf{V}(x,t) = \mathbf{V}_0,$$

where $\mathbf{V}_0$ is a constant matrix, then the monodromy matrix $\mathbf{M}(t)$ in (1.2.3) satisfies

$$\frac{d\mathbf{M}}{dt} = [\mathbf{V}_0, \mathbf{M}]. \tag{1.136}$$

Proof. Putting $x = L$ and $y = -L$ in the equation (1.134), we have

$$\frac{\partial}{\partial t}\mathbf{F}(L,-L,t) = \mathbf{V}(L,t)\mathbf{F}(L,-L,t) - \mathbf{F}(L,-L,t)\mathbf{V}(-L,t). \tag{1.137}$$

Then, we have to prove each incise separately.

(i) In the quasi-periodic case, $\mathbf{F}(L,-L,t)$ is precisely the monodromy matrix $\mathbf{M}_L(t)$. Then, multiplying (1.137) by $\mathbf{Q}$ and using the identity $\mathbf{QV}(L,t) = \mathbf{V}(-L,t)\mathbf{Q}$, we obtain (1.135).

(ii) By the Corollary 1.2.13, we have

$$\mathbf{M}(t) = \lim_{L\to\infty} \mathbf{F}(L,-L,t).$$

Taking the limit $L \to \infty$ in (1.137), we get the Lax equation (1.136).

□

The most important consequence of the time evolution for the monodromy matrix is independence of t of its trace and determinant. This result has important implications in further developments . We emphasize this point in the following theorem.

Theorem 1.4.3. *Let $\mathbf{F}(x,y)$ be the fundamental solution of (1.1), and let the monodromy matrix given by:*

- $\mathbf{M}(t) = \mathbf{F}(L,-L,t)$, *in quasi-periodic case,*

  and

- $\mathbf{M}(t) = \mathbf{F}_+^{-1}(x,t)\mathbf{F}_-(x,t)$, *in the decreasing case.*

In both cases

$$\operatorname{tr}(\mathbf{M}(t)) \quad \textit{and} \quad \det(\mathbf{M}(t))$$

are independent of t.

Finally by (1.136) and (1.139), we derive

$$\begin{aligned}
\frac{1}{\det \mathbf{M}} \frac{\mathrm{d}}{\mathrm{d}t}(\det \mathbf{M}) &= \operatorname{tr}\left(\mathbf{M}^{-1} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{M}(t))\right) \\
&= \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{V}\mathbf{M} - \mathbf{Q}\mathbf{V}\mathbf{Q}^{-1}\right) \\
&= \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{V}\mathbf{M}\right) - \operatorname{tr}\left(\mathbf{Q}\mathbf{V}\mathbf{Q}^{-1}\right) \\
&= 0.
\end{aligned}$$

□

Chapter 2

Monodromy of Linear Problems in $\mathfrak{sl}(2, \mathbb{C})$

In this Chapter we apply the general results obtained in Chapter 1 to the study of linear problems in the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ possessing the involution property. Such a class of linear problems plays an important role in the integrability theory for nonlinear evolution equations.

2.1 The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$

We introduce some algebraic facts which are necessary in further developments.

Let us consider the Lie algebra $\mathfrak{sl}(2, \mathbb{C}) \subset \mathbf{gl}(2, \mathbb{C})$ consisting of all 2×2 matrices with zero trace,

$$\mathfrak{sl}(2, \mathbb{C}) = \{\mathbf{A} \in \mathbf{gl}(2, \mathbb{C}) | \operatorname{tr} \mathbf{A} = 0\} . \tag{2.1}$$

Each element of $\mathfrak{sl}(2, \mathbb{C})$ can be written as a linear combination in some basis of $\mathfrak{sl}(2, \mathbb{C})$ [6]. The Pauli matrices defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.2}$$

form a basis of $\mathfrak{sl}(2, \mathbb{C})$. Consider also the matrices

$$\sigma_+ = \frac{\sigma_1 + i\sigma_2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \frac{\sigma_1 - i\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

It is clear that $\{\sigma_+, \sigma_-, \sigma_3\}$ is also a base of $\mathfrak{sl}(2,\mathbb{C})$.

We need the following properties of Pauli matrices:

(a) Idempotent property:

$$\sigma_i \cdot \sigma_i = \mathbf{I} \quad (i = 1,2,3). \tag{2.4}$$

(b) Commutating relations:

$$\begin{aligned} [\sigma_1, \sigma_2] &= 2i\sigma_3, \\ [\sigma_2, \sigma_3] &= 2i\sigma_1, \\ [\sigma_3, \sigma_1] &= 2i\sigma_2. \end{aligned} \tag{2.5}$$

(c) Anti-commutating relations:

$$[\sigma_i, \sigma_j]_+ := \sigma_i\sigma_j + \sigma_j\sigma_i = 0 \qquad (i,j = 1,2,3). \tag{2.6}$$

(d) Hermitian property

$$\sigma_i^* = \sigma_i \qquad (i = 1,2,3). \tag{2.7}$$

In terms of the basis $\{\sigma_+, \sigma_-, \sigma_3\}$ these properties are written as

- Commutating relations:

$$\begin{aligned} [\sigma_+, \sigma_-] &= \sigma_3, \\ [\sigma_-, \sigma_3] &= 2\sigma_-, \\ [\sigma_3, \sigma_+] &= 2\sigma_+. \end{aligned} \tag{2.8}$$

- Anti-commutating relations:

$$\begin{aligned} [\sigma_+, \sigma_-]_+ &= \mathbf{I}, \\ [\sigma_-, \sigma_3]_+ &= [\sigma_3, \sigma_+]_+ = 0, \end{aligned} \tag{2.9}$$

- σ_+ and σ_- are hermitian conjugate,

$$\sigma_+^* = \sigma_-. \tag{2.10}$$

Moreover, we have the following algebraic identities:

$$\begin{aligned} \sigma_1 \cdot \sigma_3 \cdot \sigma_1 &= -\sigma_3, \\ \sigma_1 \cdot \sigma_+ \cdot \sigma_1 &= \sigma_-, \\ \sigma_1 \cdot \sigma_- \cdot \sigma_1 &= \sigma_+, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \sigma_2 \cdot \sigma_3 \cdot \sigma_2 &= -\sigma_3, \\ \sigma_2 \cdot \sigma_+ \cdot \sigma_2 &= -\sigma_-, \\ \sigma_2 \cdot \sigma_- \cdot \sigma_2 &= -\sigma_+. \end{aligned} \tag{2.12}$$

As consequence of these identities, we deduce that

$$\sigma_i X \sigma_i \in \mathfrak{sl}(2\mathbb{C}) \quad (i = 1, 2, 3), \tag{2.13}$$

for every $X \in \mathfrak{sl}(2\mathbb{C})$. Now, we consider the Lie group

$$\mathbf{GL}(2,\mathbb{C}) = \{X \in M(n,\mathbb{C}) | \det X \neq 0\}. \tag{2.14}$$

Denote by $\mathbf{SL}(2,\mathbb{C})$ the *special linear group* consisting of all matrices with determinant equal 1,

$$\mathbf{SL}(2,\mathbb{C}) = \{\mathbf{A} \in \mathbf{GL}(2,\mathbb{C}) | \det \mathbf{A} = 1\}. \tag{2.15}$$

Then, $\mathbf{SL}(2,\mathbb{C})$ is a connected Lie subgroup of $\mathbf{GL}(2,\mathbb{C})$ [7]. It is easy to state the relationship between the Lie group $\mathbf{SL}(2,\mathbb{C})$ and the Lie Algebra $\mathfrak{sl}(2,\mathbb{C})$. Indeed, if $\mathfrak{g} \subset \mathbf{gl}(n,\mathbb{C})$ is the Lie algebra of $\mathfrak{sl}(2,\mathbb{C})$, then

$$\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C}). \tag{2.16}$$

Property of the matrix exponential function: for any $X \in \mathbf{gl}(n,\mathbb{C})$, we have

$$\det \mathrm{e}^{X} = \mathrm{e}^{\mathrm{tr}(X)}. \tag{2.17}$$

By definition, the tangent space of a Lie group G at the identity element is the Lie algebra of G. To show (2.16), consider Lie algebra of $\mathrm{SL}(n,\mathbb{C})$. Fix $X \in \mathfrak{g}$, then there exists a smooth curve $\mathbf{A}(t) \in \mathrm{SL}(n,\mathbb{C})$ such that

- $\mathbf{A}(0) = \mathbf{I}$ (identity element),
- $\frac{d}{dt}\mathbf{A}(t)\big|_{t=0} = X$.

Since $\mathbf{A}(t)$ belongs to $\mathrm{SL}(n,\mathbb{C})$, Liouvilles' formula implies that

$$\mathrm{tr}\left(\mathbf{A}^{-1}(t)\frac{d}{dt}\mathbf{A}(t)\right) = 0.$$

Evaluating at $t = 0$, we have

$$\mathrm{tr}\left(\mathbf{A}^{-1}(t)\frac{d}{dt}\mathbf{A}(t)\bigg|_{t=0}\right) = \mathrm{tr}\, X = 0.$$

It follows that $X \in \mathfrak{sl}(n,\mathbb{C})$ and therefore $\mathfrak{g} \subset \mathfrak{sl}(n,\mathbb{C})$. We only need to check that $\mathfrak{sl}(n,\mathbb{C}) \subset \mathfrak{g}$. Fix $X \in \mathfrak{sl}(n,\mathbb{C})$ and consider the smooth curve $t \mapsto \exp(tX)$. Then,

- $\exp(tX)\big|_{t=0} = \mathbf{I}$,
- $\frac{d}{dt}\exp(tX)\big|_{t=0} = X$.

We get

$$\det\exp(tX) = \exp(\mathrm{tr}(tX)) = 1.$$

We conclude that $\exp(tX)$ is a smooth curve in $\mathrm{SL}(n,\mathbb{C})$, and due to this fact $X \in \mathfrak{g}$; so $\mathfrak{sl}(n,\mathbb{C}) \subset \mathfrak{g}$.

Involution Relations

Here, follow [8], we introduce the involution relation for elements of $\mathfrak{sl}(2,\mathbb{C})$. Given $\xi, \eta, \lambda \in \mathbb{C}$, consider the matrix function $\mathbb{C} \ni \lambda \mapsto \mathcal{H}(\lambda) \in \mathfrak{sl}(2\mathbb{C})$ defined by

$$\mathcal{H}(\lambda) \stackrel{\text{def}}{=} \xi\sigma_+ + \eta\sigma_- + \frac{\lambda}{2i}\sigma_3. \tag{2.18}$$

Definition 2.1.1. *We say that $\mathcal{H}(\lambda)$ satisfies the involution property with respect to σ_i if*

$$\sigma_i\mathcal{H}(\overline{\lambda})\sigma_i = \overline{\mathcal{H}}(\lambda) \quad \forall\lambda \in \mathbb{C}, \quad i = 1,2; \tag{2.19}$$

where the σ_i are the Pauli matrices in (2.2).

Due to the property (2.13), we have that $\sigma_i\mathcal{H}(\overline{\lambda})\sigma_i \in \mathfrak{sl}(2,\mathbb{C})$ for $i = 1,2$.

Proposition 2.1.1. *Condition (2.19) holds if and only if $\xi = \epsilon\overline{\eta}$, that is,*

$$\mathcal{H}(\lambda) = \begin{pmatrix} \frac{\lambda}{2i} & \epsilon\overline{\eta} \\ \eta & -\frac{\lambda}{2i} \end{pmatrix}, \tag{2.20}$$

where λ and η are arbitrary complex numbers and

$$\epsilon = \begin{cases} 1 & if \quad \sigma = \sigma_1 \\ -1 & if \quad \sigma = \sigma_2 \end{cases} \tag{2.21}$$

Proof. The complex conjugation of the matrix function $\mathcal{H}(\lambda)$ (2.18) has the form

$$\overline{\mathcal{H}}(\lambda) = \overline{\xi}\sigma_+ + \overline{\eta}\sigma_- - \frac{\overline{\lambda}}{2i}\sigma_3. \tag{2.22}$$

Taking $\sigma = \sigma_1$, we get

$$\sigma_1\mathcal{H}(\overline{\lambda})\sigma_1 = \eta\sigma_+ + \xi\sigma_- - \frac{\overline{\lambda}}{2i}\sigma_3. \tag{2.23}$$

Assuming that equation (2.19) is true, we obtain $\xi = \overline{\eta}$. Analogously, for $\sigma = \sigma_2$, we have

$$\sigma_2\mathcal{H}(\overline{\lambda})\sigma_2 = -\eta\sigma_+ - \xi\sigma_- - \frac{\overline{\lambda}}{2i}\sigma_3. \tag{2.24}$$

Under the assumption that equation (2.19) holds, we can conclude that $\xi = -\overline{\eta}$. □

The involution property can be extended to the Lie group $\mathbf{SL}(2,\mathbb{C})$. Let $\mathbb{C} \ni \lambda \mapsto \mathcal{H}(\lambda)$ be a matrix function in $\mathfrak{sl}(2,\mathbb{C})$. We define the function $\mathbb{C} \ni \lambda \mapsto \mathbf{A}(\lambda)$ in $\mathbf{SL}(2,\mathbb{C})$ by

$$\mathbf{A}(\lambda) \stackrel{\text{def}}{=} \exp \mathcal{H}(\lambda). \tag{2.25}$$

Proposition 2.1.2. *The matrix function $\mathbf{A}(\lambda)$ (2.25) satisfies the involution relation*

$$\sigma_i \mathbf{A}(\overline{\lambda})\sigma_i = \overline{\mathbf{A}}(\lambda) \quad (i=1,2) \tag{2.26}$$

if and only if (2.19) holds for $\mathcal{H}(\lambda)$.

Proof.

$$\begin{aligned}
\sigma\mathbf{A}(\overline{\lambda})\sigma &= \sigma\exp\left(\mathcal{H}(\overline{\lambda})\right)\sigma && (2.27)\\
&= \exp\left(\sigma\mathcal{H}(\overline{\lambda})\sigma\right) && (2.28)\\
&= \exp\left(\overline{\mathcal{H}}(\lambda)\right)\\
&= \overline{\exp\left(\mathcal{H}(\lambda)\right)}\\
&= \overline{\mathbf{A}}(\lambda). && (2.29)
\end{aligned}$$

□

We can formulate an analogous result to Proposition 2.1.1 for matrices in $\mathbf{SL}(2,\mathbb{C})$ satisfying the involution relation.

Proposition 2.1.3. *Let $\lambda \mapsto \mathbf{A}(\lambda) \in \mathrm{SL}(2,\mathbb{C})$ be a function of λ satisfying the involution property (2.29). Then*

$$\mathbf{A}(\lambda) = \begin{pmatrix} a(\lambda) & \varepsilon\overline{b}(\overline{\lambda}) \\ b(\lambda) & \overline{a}(\overline{\lambda}) \end{pmatrix}, \tag{2.30}$$

where ε is defined by (2.21).

Proof. Let $\mathbf{A}(\lambda)$ be a complex valued matrix function in $\mathrm{SL}(2,\mathbb{C})$. We assume $\mathbf{A}(\lambda)$ is of the form

$$\mathbf{A}(\lambda) = \begin{pmatrix} a(\lambda) & c(\lambda) \\ b(\lambda) & d(\lambda) \end{pmatrix}.$$

Thus, the complex conjugation of $\mathbf{A}(\lambda)$ is

$$\overline{\mathbf{A}}(\lambda) = \begin{pmatrix} \overline{a}(\lambda) & \overline{c}(\lambda) \\ \overline{b}(\lambda) & \overline{d}(\lambda) \end{pmatrix}. \tag{2.31}$$

Then, we have

$$\sigma_1\mathbf{A}(\overline{\lambda})\sigma_1 = \begin{pmatrix} d(\overline{\lambda}) & b(\overline{\lambda}) \\ c(\overline{\lambda}) & a(\overline{\lambda}) \end{pmatrix}, \tag{2.32}$$

and

$$\sigma_2\mathbf{A}(\overline{\lambda})\sigma_2 = \begin{pmatrix} d(\overline{\lambda}) & -b(\overline{\lambda}) \\ -c(\overline{\lambda}) & a(\overline{\lambda}) \end{pmatrix}. \tag{2.33}$$

Substituting (2.31)-(2.33) into (2.19), we obtain

$$d(\lambda) = \overline{a}(\overline{\lambda}), \quad c(\lambda) = \varepsilon\bar{b}(\overline{\lambda}).$$

□

2.2 Linear Problems in $\mathfrak{sl}(2,\mathbb{C})$ with Involution Property

Let us consider the following linear problem

$$\frac{d\mathbf{f}}{dx} = \mathbf{U}(x,\lambda)\mathbf{f}; \tag{2.34}$$

where $\mathbf{U}(x,\lambda)$ is a matrix function of the form

$$\mathbf{U}(x,\lambda) = \sqrt{\varkappa}\,\overline{\psi}(x)\sigma_+ + \sqrt{\varkappa}\psi(x)\sigma_- + \frac{\lambda}{2i}\sigma_3. \tag{2.35}$$

Here, $\varkappa \in \mathbb{R}$, λ is a complex parameter, and ψ is a real valued complex function. We remark the matrix function $\mathbf{U}(x,\lambda)$ (2.36) can be represented as follows:

$$\mathbf{U}(x,\lambda) = \mathbf{U}_0(x) + \frac{\lambda}{2i}\sigma_3, \tag{2.36}$$

with

$$\mathbf{U}_0(x) = \sqrt{\varkappa}\begin{pmatrix} 0 & \overline{\psi}(x) \\ \psi(x) & 0 \end{pmatrix}. \tag{2.37}$$

It follows from that the matrix $\mathbf{U}(x,\lambda)$ lies in $\mathfrak{sl}(2,\mathbb{C})$ and hence $\operatorname{tr}\mathbf{U}(x,\lambda) = 0$. It is clear that $\mathbf{U}(x,\lambda)$ has the representation (2.20) with $\eta = \sqrt{\varkappa}\psi$ and $\epsilon = \operatorname{sign}(\varkappa)$. The Proposition 2.1.1 involves the matrix function $\mathbf{U}$, as function of λ, satisfies the involution property

$$\sigma\mathbf{U}(x,\overline{\lambda})\sigma = \overline{\mathbf{U}}(x,\lambda), \tag{2.38}$$

where if $\varkappa > 0$ then $\sigma = \sigma_1$, and if $\varkappa < 0$ then $\sigma = \sigma_2$.

Properties of the Fundamental solution

Let $\mathbf{T}(x,y,\lambda)$ be the fundamental solution of (2.34),

$$\frac{d}{dx}\mathbf{T} = \mathbf{U}(x,\lambda)\mathbf{T}, \tag{2.39}$$

$$\mathbf{T}|_{x=y} = \mathbf{I}. \tag{2.40}$$

According to the results of Section 1.1, $\mathbf{T}(x,y,\lambda)$ possesses the following properties:

- If ψ is a C^∞ function, bounded on $\mathbb{R}$, then $\mathbf{T}(x, y, \lambda)$ is well defined for all $(x, y) \in \mathbb{R}^2$,
- $\mathbf{T}(x, y, \lambda)$ lies in $\mathbf{SL}(2, \mathbb{C})$ (Proposition 1.1.1).

Moreover, we have the following property.

Proposition 2.2.1. *The fundamental solution* $\mathbf{T}(x, y, \lambda)$ *of linear problem (2.39) satisfies the involution property*

$$\sigma_i \mathbf{T}(x, y, \overline{\lambda})\sigma_i = \overline{\mathbf{T}}(x, y, \lambda), \quad i = 1, 2. \tag{2.41}$$

Proof. For simplicity, we take $\sigma = \sigma_i$. Condition (2.41) can be written as follows

$$\mathbf{T}(x, y, \lambda) = \sigma \overline{\mathbf{T}}(x, y, \overline{\lambda})\sigma. \tag{2.42}$$

We only need to prove that the expression in the right hand side of (2.42) is also the fundamental solution of (2.34). Differentiating, we obtain (2.4)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sigma \overline{\mathbf{T}}(x, y, \overline{\lambda})\sigma\right) &= \sigma \frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{\mathbf{T}}(x, y, \overline{\lambda})\right)\sigma, \\
&= \sigma \overline{\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{T}(x, y, \overline{\lambda})\right)}\sigma, \\
&= \sigma \overline{\mathbf{U}}(x, \overline{\lambda})\overline{\mathbf{T}}(x, y, \overline{\lambda})\sigma, \\
&= \sigma \overline{\mathbf{U}}(x, \overline{\lambda})\sigma \cdot \sigma \overline{\mathbf{T}}(x, y, \overline{\lambda})\sigma, \\
&= \mathbf{U}(x, \overline{\lambda})\sigma \overline{\mathbf{T}}(x, y, \overline{\lambda})\sigma.
\end{aligned}$$

Taking into account that

$$\sigma \overline{\mathbf{T}}(x, x, \overline{\lambda})\sigma = \sigma \overline{\mathbf{I}}\sigma = \mathbf{I},$$

we have that $\sigma \overline{\mathbf{T}}(x, y, \overline{\lambda})\sigma$ is the fundamental solution of (2.34). By uniqueness, equation (2.42) is satisfied. □

2.3 Riccati's Equation and Asymptotic Series

The fundamental solution of the linear problem in $\mathfrak{sl}(2, \mathbb{C})$ admits an special factorization.

Theorem 2.3.1. *The fundamental solution* $\mathbf{T}(x, y, \lambda)$ *of the system (2.39)-(2.40) has the following representation:*

$$\mathbf{T}(x, y, \lambda) = (\mathbf{I} + \mathbf{W}(x, \lambda)) \cdot \exp(\mathbf{Z}(x, y, \lambda)) \cdot (\mathbf{I} + \mathbf{W}(y, \lambda))^{-1}. \tag{2.43}$$

Here

- **W** *is an anti-diagonal matrix which satisfy the Riccati equation*

$$\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}x} + i\lambda\sigma_3\mathbf{W} + \mathbf{W}\mathbf{U_0}\mathbf{W} - \mathbf{U_0} = 0, \tag{2.44}$$

- **Z** *is a diagonal matrix such that*

$$\mathbf{Z}(x,y,\lambda) = \frac{(x-y)\lambda}{2i}\sigma_3 + \int_y^x \mathbf{U_0}(z)\mathbf{W}(z,\lambda)\mathrm{d}z, \tag{2.45}$$

- **W** *and* **Z** *has the following asymptotic representations as* $|\lambda| \to \infty$*:*

$$\mathbf{W}(x,\lambda) = \sum_{n=1}^{\infty} \frac{\mathbf{W}_n(x)}{\lambda^n} + O(|\lambda|^{-\infty}), \tag{2.46}$$

$$\mathbf{Z}(x,y,\lambda) = \frac{(x-y)\lambda}{2i}\sigma_3 + \sum_{n=1}^{\infty} \frac{\mathbf{Z}_n(x,y)}{\lambda^n} + O(|\lambda|^{-\infty}). \tag{2.47}$$

Proof. Differentiating (2.43), we get

$$\frac{d\mathbf{T}}{dx} = \left(\frac{d\mathbf{W}}{dx}\exp(\mathbf{Z}) + (\mathbf{I}+\mathbf{W})\exp(\mathbf{Z})\frac{\partial \mathbf{Z}}{\partial x}\right)(\mathbf{I}+\mathbf{W}(y,\lambda)^{-1}). \tag{2.48}$$

On other hand, we have

$$\mathbf{UT} = (\mathbf{U_0} + \lambda\mathbf{U_1})(\mathbf{I}+\mathbf{W}(x,\lambda))\cdot\exp(\mathbf{Z}(x,y,\lambda))\cdot(\mathbf{I}+\mathbf{W}(y,\lambda))^{-1},$$

Comparing (2.48) with (2.3), we obtain

$$\left(\frac{d\mathbf{W}}{dx}\exp(\mathbf{Z}) + (\mathbf{I}+\mathbf{W})\exp(\mathbf{Z})\frac{\partial \mathbf{Z}}{\partial x}\right) = (\mathbf{U_0} + \lambda\mathbf{U_1})(\mathbf{I}+\mathbf{W}(x,\lambda))\cdot\exp(\mathbf{Z}(x,y,\lambda)). \tag{2.49}$$

If we split the equation above into diagonal and anti-diagonal parts, then we arrive to the following equations:

$$\frac{d\mathbf{W}}{dx} + \mathbf{W}\frac{\partial \mathbf{Z}}{\partial x} = \mathbf{U}_0 + \lambda\mathbf{U}_1\mathbf{W}, \tag{2.50}$$

$$\frac{\partial \mathbf{Z}}{\partial x} = \mathbf{U}_0\mathbf{W} + \lambda\mathbf{U}_1. \tag{2.51}$$

Since the initial condition (2.40) implies $\mathbf{Z}(x,x,\lambda) = \mathbf{0}$, the equation (2.51) is equivalent to (2.45).

In order to derive the Riccati equation (2.44), we substitute (2.51) into (2.50), using the fact that $\mathbf{U}_1$ anti-commutes with $\mathbf{W}$,i.e., $\mathbf{U}_1\mathbf{W} = -\mathbf{W}\mathbf{U}_1$.

Finally, we have to get the asymptotic series for **W** and **Z**. It is enough to prove the result for **W** because equation (2.45) give the asymptotic representation for **W**. Let us assume that

$$\mathbf{W}(x,\lambda)=\sum_{n=1}^{\infty}\frac{\mathbf{W}_n(x)}{\lambda^n}.$$

Since $\mathbf{W}(x,\lambda)$ satisfies the Riccati equation (2.44), we have

$$\sum_{n=1}^{\infty}\frac{1}{\lambda^n}\frac{d\mathbf{W_n}}{dx}+\sum_{k=1}^{\infty}i\sigma_3\frac{\mathbf{W}_k}{\lambda^{k-1}}+\left(\sum_{m=1}^{\infty}\frac{\mathbf{W}_m(x)}{\lambda^m}\right)\mathbf{U}_0\left(\sum_{s=1}^{\infty}\frac{\mathbf{W}_s(x)}{\lambda^x}\right)=0. \tag{2.52}$$

Simplifying the third term in the left hand side of (2.52), we obtain

$$\begin{aligned}\left(\sum_{m=1}^{\infty}\frac{\mathbf{W}_m(x)}{\lambda^m}\right)\mathbf{U}_0\left(\sum_{s=1}^{\infty}\frac{\mathbf{W}_s(x)}{\lambda^x}\right) &= \sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{\mathbf{W}_m\mathbf{U}_0\mathbf{W}_s}{\lambda^{m+s}},\\ &= \sum_{r=1}^{\infty}\sum_{s=1}^{s=r-1}\frac{\mathbf{W}_{r+1-s}\mathbf{U}_0\mathbf{W}_s}{\lambda^{r+1}},\end{aligned}$$

where $r=m+s-1$. If we take $q=k-1$, then we can rewrite equation (2.52) as follows

$$i\sigma_3\mathbf{W}_1-\mathbf{U}_0+\frac{1}{\lambda}\left(\frac{d\mathbf{W}_1}{dx}+i\sigma_3\mathbf{W}_2\right)+\sum_{n=2}^{\infty}\frac{1}{\lambda^n}\left(\frac{d\mathbf{W}_n}{dx}+i\sigma_3\mathbf{W}_{n+1}+\sum_{k=1}^{n-1}\mathbf{W}_k\mathbf{U}_0\mathbf{W}_{n-k}\right)=0.$$

This equation holds for all $\lambda\in\mathbb{C}-\{0\}$ and all natural numbers $n\geq 1$. Hence, we have the following set of recursion relations:

$$\begin{aligned}\mathbf{W}_1(x) &= -i\sigma_3\mathbf{U}_0(x),\\ \mathbf{W}_2(x) &= i\sigma_3\frac{d\mathbf{W}_1(x)}{dx},\\ &\vdots \end{aligned} \tag{2.53}$$

$$\mathbf{W}_{n+1}(x) = i\sigma_3\left(\frac{d\mathbf{W}_n(x)}{dx}+\sum_{k=1}^{n-1}\mathbf{W}_k(x)\mathbf{U}_0(x)\mathbf{W}_{n-k}(x)\right). \tag{2.54}$$

To conclude the result, we remark the coefficients of the asymptotic series for $\mathbf{W}(x,\lambda)$ are expressed, locally, in terms of $\mathbf{U}_0$ and its derivatives. □

The following properties of the matrix $\mathbf{W}(x,\lambda)$ will be useful in the context of time evolution of monodromy.

Proposition 2.3.2. *Let* $\mathbf{W}(x,\lambda)$ *be the anti-diagonal matrix function appearing in (2.43). Then,* *(i)* $\mathbf{W}(x,\lambda)$ *satisfies the involution property*

$$\overline{\mathbf{W}}(x,\lambda) = \sigma\mathbf{W}(x,\overline{\lambda})\sigma; \tag{2.55}$$

(ii) there exists a complex valued function $w(x,\lambda)$*, differentiable in* x *and analytic in* λ*, with asymptotic expansion*

$$w(x,\lambda) = \sum_{n=1}^{\infty} \frac{w_n(x)}{\lambda^n}; \tag{2.56}$$

such that

$$\mathbf{W}(x,\lambda) = i\sqrt{\varkappa}(w(x,\lambda)\sigma_- - \overline{w}(x,\lambda)\sigma_+). \tag{2.57}$$

The terms of the asymptotic series (2.56) are given by the relations

$$\begin{aligned} w_1(x) &= \psi(x,t), \\ w_2(x) &= -i\frac{\partial\psi(x,t)}{\partial x}, \\ &\vdots \\ w_{n+1}(x) &= -i\frac{dw_n(x)}{dx} + \varkappa\overline{\psi}(x,t)\sum_{k=1}^{n-1} w_k w_{n-k}. \end{aligned}$$

Proof. (i) It follows from relations (2.11),(2.12) and the linearity, that we have only to prove the involution property for the coefficients of the asymptotic expansion for $\mathbf{W}(x,\lambda)$ (2.46). We proceed the proof by induction. First, for $n=1$, we show that the involution property holds,

$$\sigma\mathbf{W}_1(x)\sigma = \sigma(-i\sigma_3\mathbf{U}_0(x))\sigma = i\sigma_3(\sigma\mathbf{U}_0(x)\sigma) = i\sigma_3\overline{\mathbf{U}}_0(x) = \overline{\mathbf{W}}_1(x).$$

Next, for $n=2$, we have

$$\sigma\mathbf{W}_2(x)\sigma = \sigma\left(i\sigma_3\frac{d\mathbf{W}_1(x)}{dx}\right)\sigma = -i\sigma_3\left(\frac{d\overline{\mathbf{W}}_1(x)}{dx}\right) = \overline{\mathbf{W}}_2(x).$$

Finally, we assume that $\mathbf{W}_k$ satisfies the involution property for all positive integer $k \leq n$, and prove that $\mathbf{W}_{k+1}$ satisfies (2.55). According to recursion relations (2.54)

$$\mathbf{W}_{n+1}(x) = i\sigma_3\left(\frac{d\mathbf{W}_n(x)}{dx} + \sum_{k=1}^{n-1}\mathbf{W}_k(x)\mathbf{U}_0(x)\mathbf{W}_{n-k}(x)\right),$$

we have

$$\begin{aligned}\sigma_i \mathbf{W}_{n+1}(x)\sigma_i &= -i\sigma_3\left(\frac{d\overline{\mathbf{W}}_n(x)}{dx}+\sum_{k=1}^{n-1}\sigma_i\mathbf{W}_k(x)\sigma_i\sigma_i\mathbf{U}_0(x)\sigma_i\sigma_i\mathbf{W}_{n-k}(x)\sigma_i\right),\\ &= -i\sigma_3\left(\frac{d\overline{\mathbf{W}}_n(x)}{dx}+\sum_{k=1}^{n-1}\overline{\mathbf{W}}_k(x)\overline{\mathbf{U}}_0(x)\overline{\mathbf{W}}_{n-k}(x)\right),\\ &= \overline{\mathbf{W}}_{n+1}.\end{aligned}$$

The matrix function $\mathbf{W}(x,\lambda)$ satisfies the involution relation because all the coefficients of its asymptotic series have this property.

(ii) Since the matrix function $\mathbf{W}(x,\lambda)$ satisfies the involution property, Proposition 2.1.3 implies the existence of the function $w(x)$. The asymptotic series for $w(x)$ follows directly from the asymptotic representation of $\mathbf{W}$ (Proposition 2.3.2). The equation for the coefficients $w_n(x)$ are derived from recursion relations (2.54). Differentiability and analyticity for the matrix $\mathbf{U}(x,\lambda)$ are consequence of the representation (2.4).

□

2.4 Representation for the Monodromy in the Quasi-periodic case

We return to the linear problem (2.39)-(2.40). Assume that $\mathbf{U}(x,\lambda)$ (2.4) satisfy the quasi-periodicity condition

$$\mathbf{U}(x+2L,\lambda) = \mathbf{Q}^{-1}(\theta)\mathbf{U}(x,\lambda)\mathbf{Q}(\theta),) \tag{2.58}$$

with

$$\mathbf{Q}(\theta) = \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}); \tag{2.59}$$

where $\theta \in \mathbb{R}$ is a constant. The quasi-periodic property for $\mathbf{U}(x,\lambda) \in \mathfrak{sl}(2,\mathbb{C})$ is controlled by off-diagonal components $\psi(x)$ and $\overline{\psi}(x)$ in (2.36),(2.37).

Proposition 2.4.1. *The quasi-periodic condition (2.58) for the matrix valued function* $\mathbf{U}(x,\lambda)$ *with* $\mathbf{Q}$ *in (2.59) is equivalent to the following condition for* ψ*:*

$$\psi(x+2L) = \psi(x)e^{i\theta}. \tag{2.60}$$

Proof. Using that σ_3 and $\mathbf{Q}$ commute, we have

$$\begin{aligned}
\mathbf{Q}^{-1}(\theta)(\mathbf{U}(x))\mathbf{Q}(\theta) &= \begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix} \begin{pmatrix} \frac{\lambda}{2i} & \sqrt{\varkappa}\overline{\psi}(x) \\ \sqrt{\varkappa}\psi(x) & -\frac{\lambda}{2i} \end{pmatrix} \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix}, \\
&= \begin{pmatrix} \frac{\lambda}{2i} & \sqrt{\varkappa}\overline{\psi}(x)e^{-i\theta} \\ \sqrt{\varkappa}\psi(x)e^{i\theta} & -\frac{\lambda}{2i} \end{pmatrix}, \\
&= \sqrt{\varkappa}e^{-\frac{i\theta}{2}}\overline{\psi}(x)\sigma_+ + \sqrt{\varkappa}e^{\frac{i\theta}{2}}\psi(x)\sigma_- + \frac{\lambda}{2i}\sigma_3.
\end{aligned}$$

From here, we deduce the desired result. □

This motivates the next definition.

Definition 2.4.1. *We say that a complex valued function $u(x)$ which depends on $x \in \mathbb{R}$ satisfies the quasi-periodicity condition, if there exist a real number $L > 0$ and $0 \le \theta < 2\pi$ such that $u(x+2L) = e^{i\theta}u(x)$*

Recall that the fundamental solution $\mathbf{T}(x, y, \lambda)$ admits the factorizations (Theorem 2.3.1)

$$\mathbf{T}(x, y, \lambda) = (\mathbf{I} + \mathbf{W}(x, \lambda)) \cdot \exp(\mathbf{Z}(x, y, \lambda)) \cdot (\mathbf{I} + \mathbf{W}(y, \lambda))^{-1}.$$

Taking into account the properties of the matrix factors $\mathbf{W}(x, \lambda)$ (Theorem 2.3.1 and Proposition 2.3.2), we derive the following result.

Proposition 2.4.2. *The anti-diagonal matrix function $\mathbf{W}(x, \lambda)$ is expressed by the formula*

$$\mathbf{W}(x, \lambda) = i\sqrt{\varkappa}(w(x, \lambda)\sigma_- - \overline{w}(x, \lambda)\sigma_+), \tag{2.61}$$

where w satisfies the quasi-periodicity condition:

$$w(x + 2L, \lambda) = e^{i\theta}w(x, \lambda), \tag{2.62}$$

and hence

$$\mathbf{W}(x + 2L, \lambda) = \mathbf{Q}^{-1}\mathbf{W}(x, \lambda)\mathbf{Q},$$

with $\mathbf{Q}$ of the form (2.59)

Proof. By Proposition 2.3.2, the matrix $\mathbf{W}(x, \lambda)$ has asymptotic representation. Therefore, we only need to prove the quasi-periodic condition for the coefficients $\mathbf{W}_n(x, \lambda)$ of the asymptotic expansion of $\mathbf{W}(x, \lambda)$ (2.46). We proceed the proof by induction. First, we take $n = 1$. To prove that $\mathbf{W}_1$ is quasi-periodic, we note thatboth σ_3 and

$\mathbf{Q} = \exp\{\frac{i\theta}{2}\sigma_3\}$ are diagonal matrices, so that they commute. Since the matrix function $\mathbf{U}(x,\lambda)$ (2.36) satisfies the quasi-periodic condition, $\mathbf{U}_0(x)$ does it. Using the recursion relation for the the asymptotic series of $\mathbf{W}(x,\lambda)$ (2.54), we get

$$\mathbf{W}_1(x+2L) = -i\sigma_3\mathbf{U}_0(x+2L) = \mathbf{Q}^{-1}(-i\sigma_3\mathbf{U}_0(x))\mathbf{Q} = \mathbf{Q}^{-1}\mathbf{W}_1(x)\mathbf{Q}.$$

For $n=2$, recursion relations (2.54) imply

$$\mathbf{W}_2(x+2L) = i\sigma_3\frac{d\mathbf{W}_1(x+2L)}{dx} = \mathbf{Q}^{-1}(i\sigma_3\frac{d\mathbf{W}_1(x)}{dx})\mathbf{Q} = \mathbf{Q}^{-1}\mathbf{W}_2(x)\mathbf{Q}.$$

Finally, assume that the coefficient $\mathbf{W}_k(x)$ of the asymptotic series has the quasi-periodic condition for all positive integer $k \leq n$. Again by the recursion relation (2.54), we conclude that the coefficient $\mathbf{W}_{n+1}(x)$ has the desired condition. The second part of the Proposition follows immediately from Proposition 2.4.1. □

Properties of the Monodromy

The monodromy matrix (Definition 1.2.2) for the linear system (2.34) in the quasi-periodic case (2.58),with matrix coefficient $\mathbf{U}(x,\lambda)$ satisfying the involution property (2.19), is

$$\mathbf{M}(\lambda) = \mathbf{T}(L,-L,\lambda), \tag{2.63}$$

where $\mathbf{T}(x,y,\lambda)$ is the fundamental solution. The monodromy matrix enjoys of the general properties, established in Proposition 1.2.6. Within the context of linear system in $\mathfrak{sl}(2,\mathbb{C})$ with the involution property, It is possible to derive additional properties for the monodromy matrix.

The first one is presented in the next Proposition.

Proposition 2.4.3. *The monodromy matrix* $\mathbf{M}(\lambda)$ *has the following representation*

$$\mathbf{M}(\lambda) = (\mathbf{I}+\mathbf{W}(L,\lambda))\exp(\mathbf{Z}_L(\lambda))(\mathbf{I}+\mathbf{W}(-L,\lambda))^{-1}, \tag{2.64}$$

where

$$\mathbf{Z}_L(\lambda) = -i\lambda L\sigma_3 + \int_{-L}^{L}\mathbf{U}_0\mathbf{W}(x,\lambda)dx, \tag{2.65}$$

and $\mathbf{W}(x,\lambda)$ *is defined by (2.44). (2.3.1).*

Proof. By Theorem 2.3.1, the fundamental solution is of the form (2.43) for all $x, y \in \mathbb{R}$. In particular, it is valid for fundamental solutions at $x = L$,$y = -L$, which coincides with the monodromy matrix. □

Here is another property of the monodromy matrix.

Proposition 2.4.4. *The monodromy matrix satisfies the involution property (2.19) and is of the form*

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & \varepsilon\bar{b}(\bar{\lambda}) \\ b(\lambda) & \bar{a}(\bar{\lambda}) \end{pmatrix}, \tag{2.66}$$

where

$$\epsilon = \begin{cases} 1 & \text{if } \varkappa > 0 \\ -1 & \text{if } \varkappa < 0 \end{cases} \tag{2.67}$$

Proof. Since the fundamental solution satisfies the involution property for all $x, y \in \mathbb{R}$, the monodromy so does. As consequence of this property, and Proposition 2.1.3 we get (2.66). □

The complex functions $a(\lambda)$ and $b(\lambda)$ are called the *transition coefficients* of the linear problem (2.34) [8]. Finally, we establish the main properties of the transition coefficients.

Proposition 2.4.5. *The transition coefficients $a(\lambda)$ and $b(\lambda)$ possesses the following properties:*

(i) $a(\lambda)$ and $b(\lambda)$ are analytic function in $\lambda \in \mathbb{C}$,

(ii) for each real λ, the normalization condition holds

$$|a(\lambda)|^2 - \varepsilon|b(\lambda)|^2 = 1; \tag{2.68}$$

(iii) $a(\lambda)$ and $b(\lambda)$ are given by

$$a(\lambda) = \frac{e^{i\alpha(\lambda)} - \varkappa w(-L,\lambda)\overline{w}(L,\bar{\lambda})e^{-i\alpha(\lambda)}}{1 - \varkappa w(-L,\lambda)\overline{w}(-L,\bar{\lambda})}, \tag{2.69}$$

$$b(\lambda) = \frac{i\sqrt{\varkappa}(e^{i\alpha(\lambda)}w(L,\lambda) - w(L,\lambda)e^{-i\alpha(\lambda)})}{1 - \varkappa w(-L,\lambda)\overline{w}(-L,\bar{\lambda})}; \tag{2.70}$$

where

$$\alpha(\lambda) = -L\lambda + \varkappa\int_{-L}^{L}\bar{(\psi)}(x)w(x,\lambda)dx. \tag{2.71}$$

Proof. (i) The analyticity of $\mathbf{M}(\lambda)$ is provided by the analytic properties of the fundamental solution $\mathbf{T}(x,y,\lambda)$.

(ii) Since $\mathbf{M}(\lambda)$ belongs to $\mathbf{SL}(2,\mathbb{C})$, we have $\det\mathbf{M} = a(\lambda)\bar{a}(\bar{\lambda}) - \varepsilon b(\lambda)\bar{b}(\bar{\lambda}) = 1$. Considering only real values for λ, we obtain the normalization relation.

(iii) Starting from the factorization of the monodromy matrix $\mathbf{M}(\lambda)$ (2.4.3)

$$\mathbf{M}(\lambda) = (\mathbf{I} + \mathbf{W}(L,\lambda))\exp(\mathbf{Z}_L(\lambda))(\mathbf{I} + \mathbf{W}(-L,\lambda))^{-1},$$

we derive

$$\mathbf{M}(\lambda)(\mathbf{I} + \mathbf{W}(-L,\lambda)) = (\mathbf{I} + \mathbf{W}(L,\lambda))\exp(\mathbf{Z}_L(\lambda)). \tag{2.72}$$

By equation (2.72), we get the set of equations for the matricial components

$$\begin{aligned} a(\lambda) + i\varepsilon\sqrt{\varkappa}w(-L,\lambda)\bar{b}(\bar{\lambda}) &= e^{i\alpha(\lambda)}, \\ -i\sqrt{\varkappa}\overline{w}(-L,\bar{\lambda})a(\lambda) + \varepsilon\bar{b}(\bar{\lambda}) &= -i\sqrt{\varkappa}\overline{w}(L,\bar{\lambda})e^{-i\alpha(\lambda)}, \\ i\sqrt{\varkappa}w(-L,\lambda)\bar{a}(\bar{\lambda}) &= i\sqrt{\varkappa}w(L,\lambda)e^{i\alpha(\lambda)}, \\ \bar{a}(\bar{\lambda}) - i\sqrt{\varkappa}\overline{w}(-L,\bar{\lambda})b(\lambda) &= e^{-i\alpha(\lambda)}. \end{aligned}$$

Resolving these equations relative to $a(\lambda)$ and $b(\lambda)$, we obtain the representations (2.69), (2.70) for the transition coefficients.

□

The time evolution and Integrals of motion

In Section 1.4 of Chapter 1, we studied the time evolution of the monodromy matrix when the matrix coefficient $\mathbf{U}(x)$ of the linear system with monodromy $\mathbf{M}(t)$ together with another matrix $\mathbf{V}(x)$ are solutions of the zero curvature equation (1.33). We conclude that the determinant and the trace of the monodromy matrix $\mathbf{M}(t)$ are time invariants. Hence, they are the integrals of motion for the zero curvature equation.

In this part, the purpose is to write these time-invariants in terms of the matrix coefficient $\mathbf{U}(x,\lambda)$ (2.36) of the linear system in $\mathfrak{sl}(2,\mathbb{C})$ and the transition coefficients of the monodromy.

Let us define the function $F_L : \mathbb{C} \to \mathbb{C}$ by

$$F_L(\lambda) = \operatorname{tr}\mathbf{M}(\lambda)\mathbf{Q}(\theta). \tag{2.73}$$

Follow [8], the function 2.73 will be called *generation function of the conservation laws.* Here, we shall prove the following results.

Proposition 2.4.6. *The generating function is expressed in terms of the transition coefficients $a(\lambda)$, $b(\lambda)$ as follows 2.4.4) by*

$$F(\lambda) = a(\lambda)e^{\frac{i\theta}{2}} + \bar{a}(\bar{\lambda})e^{\frac{-i\theta}{2}}. \tag{2.74}$$

The proof follows directly from the formula

$$\mathbf{M}(\lambda)\mathbf{Q} = \begin{pmatrix} a(\lambda) & \varepsilon\bar{b}(\bar{\lambda}) \\ b(\lambda) & \bar{a}(\bar{\lambda}) \end{pmatrix} \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{-i\theta}{2}} \end{pmatrix}.$$

Theorem 2.4.7. *The generating function $F(\lambda)$ (2.73) has the form*

$$F_L(\lambda) = 2\cos(\varphi_L(\lambda) + \frac{\theta}{2} - \lambda L), \tag{2.75}$$

where $\varphi_L(\lambda)$ is given by

$$\varphi_L(\lambda) = \varkappa \int_{-L}^{L} \overline{\psi}(x,t) w(x,\lambda) dx. \tag{2.76}$$

$F(\lambda)$ has the following properties:

(i) $\varphi_L(\lambda) = \overline{\varphi}_L(\overline{\lambda})$;

(ii) $\varphi_L(\lambda) = \varkappa \sum_{n=1}^{\infty} \frac{I_n}{\lambda^n} + O(|\lambda|^{-\infty})$ with

$$I_n = \int_{-L}^{L} \overline{\psi}(x,t) w_n(x). \tag{2.77}$$

Proof. Since the matrix function $\mathbf{W}(x,\lambda)$ is quasi-periodic, the matrix $(\mathbf{I} + \mathbf{W}(x,\lambda))$ and its inverse are also quasi-periodic,

$$\begin{aligned} \mathbf{I} + \mathbf{W}(x+2L,\lambda) &= \mathbf{Q}^{-1}(\theta)(\mathbf{I} + \mathbf{W}(x,\lambda))\mathbf{Q}(\theta), \\ (\mathbf{I} + \mathbf{W}(x+2L,\lambda))^{-1} &= \mathbf{Q}^{-1}(\theta)\,(\mathbf{I} + \mathbf{W}(x,\lambda))^{-1}\,\mathbf{Q}(\theta). \end{aligned}$$

Using this fact and Theorem 2.4.3, we get

$$\mathbf{M}(\lambda)\mathbf{Q}(\theta) = (\mathbf{I} + \mathbf{W}(L,\lambda))\exp(\mathbf{Z}_L(\lambda))\mathbf{Q}(\theta)(\mathbf{I} + \mathbf{W}(L,\lambda))^{-1}, \tag{2.78}$$

where

$$\mathbf{Z}_L(\lambda) = -i\lambda L\sigma_3 + \int_{-L}^{L} \mathbf{U}_0 \mathbf{W}(x,\lambda) dx,$$

and

$$\mathbf{U}_0(x)\mathbf{W}(x,\lambda) = i\varkappa \begin{pmatrix} \overline{\psi}(x) w(x,\lambda) & 0 \\ 0 & -\psi(x)\overline{w}(x,\overline{\lambda}) \end{pmatrix}. \tag{2.79}$$

Consider $\varphi_L(\lambda)$ defined by (2.76) From the commutativity of σ_3 and $\mathbf{Z}_L(\lambda)$, we get

$$F_L(\lambda) = \operatorname{tr} \mathbf{M}(\lambda)\mathbf{Q}(\theta) = \operatorname{tr} \exp\left\{ \mathbf{Z}_L(\lambda) + \frac{i\theta}{2}\sigma_3 \right\}. \tag{2.80}$$

Since $\mathbf{M}(\lambda)\mathbf{Q}(\theta)$ has determinant 1, it follows that

$$\operatorname{tr} \mathbf{Z}_L(\lambda) = O(|\lambda|^{-\infty}). \tag{2.81}$$

Thus, the coefficients of the matrix function

$$\varkappa \int_{-L}^{L} \mathbf{U}_0(x)\mathbf{W}(x,\lambda)dx$$

are real valued functions, and

$$\varphi_L(\lambda) = \overline{\varphi}_L(\overline{\lambda}). \tag{2.82}$$

Next, we have

$$\mathbf{Z}_L(\lambda) = i\sigma_3(\varphi_L(\lambda) - \lambda L), \tag{2.83}$$

and from (2.80) we derive

$$\begin{aligned} F_L(\lambda) &= \operatorname{tr}\exp\left((\varphi_L(\lambda) + \frac{\theta}{2} - \lambda L)i\sigma_3\right) \\ &= \mathrm{e}^{i\left(\varphi_L(\lambda)+\frac{\theta}{2}-\lambda L\right)} + \mathrm{e}^{-i\left(\varphi_L(\lambda)+\frac{\theta}{2}-\lambda L\right)} \\ &= 2\cos\left(\varphi_L(\lambda) + \frac{\theta}{2} - \lambda L\right). \end{aligned}$$

Finally, taking into account that the function $w(x,\lambda)$ has the asymptotic series (2.56), we obtain

$$\varphi_L(\lambda) = \varkappa \sum_{n=1}^{\infty} \frac{I_n}{\lambda^n} + O(|\lambda|^{-\infty}), \tag{2.84}$$

with

$$I_n(\psi, \overline{\psi}) = \int_{-L}^{L} \overline{\psi}(x) w_n(x). \tag{2.85}$$

□

For all $n = 1, 2, \dots$, the functionals I_n (2.85) are the local integrals of motion the quasi-periodic system. Using the formulas for the coefficients of the asymptotic expansion of $w(x)$ (2.56), we can express the first four integrals of motion in terms of the function $\psi(x)$:

$$\begin{aligned} I_1 &= \int_{-L}^{L} |\psi(x)|^2 dx, \\ I_2 &= \int_{-L}^{L} \left(-i\bar{\psi}(x)\frac{d}{dx}\psi(x)\right) dx, \\ I_3 &= \int_{-L}^{L} \left(-\bar{\psi}(x)\frac{d^2}{dx^2}\psi(x) + \varkappa|\psi(x)|^4\right) dx, \\ I_4 &= \int_{-L}^{L} i\left(\bar{\psi}(x)\frac{d^3}{dx^3}\psi(x) - \varkappa|\psi(x)|^2\left(\psi(x)\frac{d}{dx}\bar{\psi}(x) + 4\bar{\psi}(x)\frac{d}{dx}\psi(x)\right)\right) dx. \end{aligned}$$

2.5 Representation for Monodromy in the Decreasing Case.

We shall apply the results of Chapter 1 and derive additional properties for the fundamental solutions and the monodromy matrix of the linear system on $\mathfrak{sl}(2,\mathbb{C})$ with boundary conditions in the decreasing case.

Consider the linear problem

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}x} = \mathbf{U}(x,\lambda)\mathbf{f},$$

with matrix coefficient

$$\mathbf{U}(x,\lambda) = \mathbf{U}_0(x) + \frac{\lambda}{2i}\sigma_3, \tag{2.86}$$

where $\mathbf{U}_0(x)$ is of the form (2.37)and lies on $L_1^{2\times 2}(\mathbb{R})\bigcap C^\infty(\mathbb{R})$. We assume that the parameter λ in (2.86) is real . To get some special properties of the monodormy, the idea is to study the analytic continuation of the fundamental solution in the complex plane $\lambda \in \mathbb{C}$. The integrability assumption for $\mathbf{U}_0(x)$ implies that $\psi \in L_1(\mathbb{R})$. Moreover, this is a sufficient condition for the existence of the fundamental solution $\mathbf{T}(x,y,\lambda)$ of the linear system (2.5) satisfying (2.39)-(2.40) [14].

In order to define the monodromy matrix in the decreasing case, we need to use the following functions

$$\mathbf{T}_\pm(x,\lambda) = \lim_{y\to\pm\infty} \mathbf{T}(x,y,\lambda)\mathbf{E}(y,\lambda). \tag{2.87}$$

Here, the matrix function

$$\mathbf{E}(x,\lambda) = \exp\left(\frac{\lambda}{2i}x\sigma_3\right) \tag{2.88}$$

is the fundamental solution of the linear problem

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{E}(x,\lambda) &= \frac{\lambda}{2i}\sigma_3\mathbf{E}(x,\lambda) \\ \mathbf{E}(0,\lambda) &= \mathbf{I}, \quad \forall\lambda\in\mathbb{R}. \end{aligned}$$

In Chapter 1, we have derived the following properties of $\mathbf{T}_\pm(x,\lambda)$:

- for each x and real λ, $\mathbf{T}_\pm(x,\lambda)$ are well defined (Proposition 1.3.5);
- $\mathbf{T}_\pm(x,\lambda)$ have the integral representations:

$$\mathbf{T}_-(x,\lambda) = \mathbf{E}(x,\lambda) + \int_{-\infty}^{x} \mathbf{E}(x-z,\lambda)\mathbf{U}_0(z)\mathbf{T}_-(z,\lambda)\mathrm{d}z, \tag{2.89}$$

$$\mathbf{T}_+(x,\lambda) = \mathbf{E}(x,\lambda) - \int_{x}^{\infty} \mathbf{E}(x-z,\lambda)\mathbf{U}_0(z)\mathbf{T}_+(z,\lambda)\mathrm{d}z; \tag{2.90}$$

(Proposition 1.2.10),

- $\mathbf{T}_\pm(x,\lambda)$ satisfy the linear problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{T}_\pm = \mathbf{U}(x,\lambda)\mathbf{T}_\pm, \tag{2.91}$$

with the asymptotic conditions (Proposition 1.3.5)

$$\mathbf{T}_\pm(x,\lambda) \to \mathbf{E}(x,\lambda) \quad \text{as} \quad x \to \pm\infty. \tag{2.92}$$

Follow [8], the functions $\mathbf{T}_\pm(x,\lambda)$ will be called *Jost solutions*. They will play an important role in the further developments. Let us introduce the following notation for the Jost solutions:

$$\mathbf{T}_\pm(x,\lambda) = (\mathbf{T}_\pm^{(1)}(x,\lambda), \mathbf{T}_\pm^{(2)}(x,\lambda)), \tag{2.93}$$

where $\mathbf{T}_\pm^{(i)}(x,\lambda)$, i=1,2, are column vectors. If the matrix coefficient $\mathbf{U}(x,\lambda)$ is analytic in λ, then $\mathbf{T}(x,y,\lambda)$ is an entire function. However, $\mathbf{T}_\pm(x,\lambda)$ are not analytic, in general, because the limit in (2.87) does not exit for any λ with $\mathrm{Im}\lambda \neq 0$. Although, each column of the $\mathbf{T}_\pm(x,\lambda)$ has analytic continuation over some complex domain.

Proposition 2.5.1. *Let $\mathbf{T}_\pm(x,\lambda)$ be the Jost functions defined by (2.87). Then,*

(a) the vector functions $\mathbf{T}_-^{(1)}(x,\lambda)$ and $\mathbf{T}_+^{(2)}(x,\lambda)$ are analytic on the upper half plane in $\mathbb{C}$,

b) the vector functions $\mathbf{T}_+^{(1)}(x,\lambda)$ and $\mathbf{T}_-^{(2)}(x,\lambda)$ are analytic on the lower half plane in $\mathbb{C}$.

Proof. First, let us find the domains of analyticity for the columns of $\mathbf{T}_+(x,\lambda)$. In vector notation, the formulas for the Jost solution $\mathbf{T}_+(x,\lambda)$ read

$$\mathbf{T}_+^{(1)}(x,\lambda) = \lim_{y\to\infty} \mathrm{e}^{-\frac{y\lambda}{2}}\mathbf{T}^{(1)}(x,y,\lambda), \tag{2.94}$$

$$\mathbf{T}_+^{(2)}(x,\lambda) = \lim_{y\to\infty} \mathrm{e}^{\frac{y\lambda}{2}}\mathbf{T}^{(2)}(x,y,\lambda). \tag{2.95}$$

Then, we can see that the limit in (2.94) exists for every $\lambda \in \mathbb{C}$ with $\mathrm{Im}\lambda \leq 0$ because the factor $\mathrm{e}^{-\frac{y\lambda}{2}}\mathbf{T}^{(1)}(x,y,\lambda)$ decays if $y \to \infty$. Analogously, the limit in (2.95) exists for $\lambda \in \mathbb{C}$ with $\mathrm{Im}\lambda \geq 0$ because the factor $\mathrm{e}^{\frac{y\lambda}{2}}\mathbf{T}^{(2)}(x,y,\lambda)$ also decays. These limits define analytic functions in the upper and lower complex planes, respectively, because the functions $\mathrm{e}^{-\frac{y\lambda}{2}}\mathbf{T}^{(1)}(x,y,\lambda)$ and $\mathrm{e}^{\frac{y\lambda}{2}}\mathbf{T}^{(2)}(x,y,\lambda)$ are analytic functions. Therefore, we get the analytic continuation of $\mathbf{T}_+^{(1)}(x,\lambda)$ and $\mathbf{T}_+^{(2)}(x,\lambda)$.

Finally, in the same way, we proof the items (a) and (b) for the columns of the Jost solution $\mathbf{T}_-(x,\lambda)$. □

Properties of the Monodromy

In the decreasing case, the monodromy matrix is defined by

$$\mathbf{M}(\lambda) = \mathbf{T}_+^{-1}(x,\lambda)\mathbf{T}_-(x,\lambda), \text{ for all real } \lambda. \tag{2.96}$$

As it was seen in the Section (1.2.3), the mondromy matrix enjoys the following properties

- $\mathbf{M}(\lambda)$ does not depend on x,
- $\mathbf{T}(x,y,\lambda) = \mathbf{T}_+(x,\lambda)\mathbf{M}(\lambda)\mathbf{T}_-^{-1}(y,\lambda)$, and
- $\mathbf{M}(\lambda) = \lim_{L\to\infty} \mathbf{T}_+^{-1}(L,\lambda)\mathbf{T}(L,-L,\lambda)\mathbf{T}_-(-L,\lambda)$.

The last ones are general properties for the monodromy in the decreasing case. For the linear system in $\mathfrak{sl}(2,\mathbb{C})$, we have the next additional properties.

Proposition 2.5.2. *Let* $\mathbf{M}(\lambda)$ *be the monodromy matrix of the linear system (2.5),(2.86) in* $\mathfrak{sl}(2,\mathbb{C})$. *The following properties hold:*

(i) involution property,

$$\overline{\mathbf{M}}(\lambda) = \sigma\mathbf{M}(\lambda)\sigma, \qquad \sigma = \sigma_i \tag{2.97}$$

(ii) there exist complex valued functions $a(\lambda)$,$b(\lambda)$ *such that*

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & \epsilon\bar{b}(\lambda) \\ b(\lambda) & \overline{a}(\lambda) \end{pmatrix}, \tag{2.98}$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } \varkappa > 0, \\ -1 & \text{if } \varkappa < 0, \end{cases} \tag{2.99}$$

and

$$|a(\lambda)|^2 - \epsilon|b(\lambda)|^2 = 1. \tag{2.100}$$

Proof. (i) Let us check that $\mathbf{T}_\pm(x,\lambda)$ satisfy (2.97). Since $\mathbf{T}(x,y,\lambda)\mathbf{E}(y,\lambda)$ converges uniformly to $\mathbf{T}_\pm(x,\lambda)$ as $y \to \pm\infty$, we need to verify the involution property for $\mathbf{T}(x,y,\lambda)\mathbf{E}(y,\lambda)$. This is just straightforward computation

$$\begin{aligned} \sigma\mathbf{T}(x,y,\lambda)\mathbf{E}(y,\lambda)\sigma &= \sigma\mathbf{T}(x,y,\lambda)\sigma\sigma\mathbf{E}(y,\lambda)\sigma, \\ &= \overline{\mathbf{T}}(x,y,\lambda)\exp\left\{\frac{\lambda}{2i}\sigma\sigma_3\sigma\right\}, \\ &= \overline{\mathbf{T}}(x,y,\lambda)\exp\left\{\frac{-\lambda}{2i}\sigma_3\right\}, \\ &= \overline{(\mathbf{T}(x,y,\lambda)\mathbf{E}(y,\lambda))}. \end{aligned}$$

Since $\mathbf{T}_{\pm}(x,\lambda)$ has the involution property, we have

$$\begin{aligned}\overline{\mathbf{M}}(\lambda) &= \overline{(\mathbf{T}_{+}^{-1}(x,\lambda)\mathbf{T}_{-}(x,\lambda))},\\ &= (\overline{\mathbf{T}}_{+}(x,\lambda))^{-1}\overline{\mathbf{T}}_{-}(x,\lambda),\\ &= (\sigma\mathbf{T}_{+}^{-1}(x,\lambda)\sigma)\sigma\mathbf{T}_{-}(x,\lambda)\sigma,\\ &= \sigma\mathbf{M}(\lambda)\sigma.\end{aligned}$$

(ii) Lemma 2.1.3 and the involution property (2.97) of $\mathbf{M}(\lambda)$ imply the existence of the complex valued functions $a(\lambda), b(\lambda)$. These functions satisfy the normalized relation (2.100) because of the unimodularity of the monodromy.

□

Trace formulas for monodromy. We shall derive the local integrals of motion and derive formulae for the transition coefficients. First, we recall that the matrix coefficient of linear system (2.5) has the form

$$\mathbf{U}(x,\lambda) = \sqrt{\varkappa}\,\overline{\psi}(x)\sigma_{+} + \sqrt{\varkappa}\psi(x)\sigma_{-} + \frac{\lambda}{2i}\sigma_3.$$

Proposition 2.5.3. *The trace of the monodromy matrix is given by*

$$\operatorname{tr}\mathbf{M}(\lambda) = 2\mathrm{Re}a(\lambda) = 2|a(\lambda)|\cos(\arg a(\lambda)). \tag{2.101}$$

The proof is a straightforward consequence of Proposition 2.5.2. Our next goal is to derive a formula for the trace of the monodromy in term of the integrable function $\psi(x)$ appearing in the matrix coefficient $\mathbf{U}(x)$ of the linear system (2.5).

Proposition 2.5.4. *The trace of the monodromy matrix $\mathbf{M}(\lambda)$ satisfy*

$$\operatorname{tr}\mathbf{M}(\lambda) = 2\cos\left(\varkappa\int_{-\infty}^{\infty}\overline{\psi}(x)w(x,\lambda)dx\right). \tag{2.102}$$

Proof. By definition of monodromy, we have

$$\mathbf{M}(\lambda) = \mathbf{T}_{+}^{-1}(x,\lambda)\mathbf{T}_{-}(x,\lambda),$$

where

$$\mathbf{T}_{\pm}(x,\lambda) = \lim_{y\to\pm\infty}\mathbf{T}(x,y\lambda)\mathbf{E}(y,\lambda).$$

On other hand, the fundamental solution has the factorization (Theorem 2.3.1)

$$\mathbf{T}(x,y,\lambda) = (\mathbf{I}+\mathbf{W}(x,\lambda))\cdot\exp(\mathbf{Z}(x,y,\lambda))\cdot(\mathbf{I}+\mathbf{W}(y,\lambda))^{-1}.$$

Now, we only need to expresses the monodromy matrix in terms of the matrix factors for fundamental solution. First, the Jost solution have the form

$$\mathbf{T}_-(x,\lambda) = (\mathbf{I}+\mathbf{W}(x,\lambda))\exp\left(\frac{x\lambda}{2i}+\int_{-\infty}^{x}\mathbf{U}_0(z)\mathbf{W}(z,\lambda)dz\right)\lim_{y\to-\infty}\boldsymbol{\Upsilon}(y,\lambda),$$

$$\mathbf{T}_+(x,\lambda) = (\mathbf{I}+\mathbf{W}(x,\lambda))\exp\left(\frac{x\lambda}{2i}-\int_{x}^{\infty}\mathbf{U}_0(z)\mathbf{W}(z,\lambda)dz\right)\lim_{y\to-\infty}\boldsymbol{\Upsilon}(-y,\lambda),$$

where

$$\boldsymbol{\Upsilon}(y,\lambda) \stackrel{\text{def}}{=} \mathbf{E}(-y,\lambda)\,(\mathbf{I}+\mathbf{W}(y,\lambda))^{-1}\,\mathbf{E}(y,\lambda).$$

Then, for the monodromy matrix we get

$$\mathbf{M}(\lambda) = \lim_{y\to-\infty}\boldsymbol{\Upsilon}^{-1}(y,\lambda)\exp\left(\int_{-\infty}^{\infty}\mathbf{U}_0(z)\mathbf{W}(z,\lambda)dz\right)\lim_{y\to-\infty}\boldsymbol{\Upsilon}(y,\lambda).$$

Therefore,

$$\operatorname{tr}\mathbf{M}(\lambda) = \operatorname{tr}\exp\left(\int_{-\infty}^{\infty}\mathbf{U}_0(z)\mathbf{W}(z,\lambda)dz\right).$$

Since the components of the diagonal matrix $i\int_{-\infty}^{\infty}\mathbf{U}_0(z)\mathbf{W}(z,\lambda)$ are real valued functions, we obtain

$$\operatorname{tr}\mathbf{M}(\lambda) = \operatorname{tr}\exp\left(i\varkappa\int_{-\infty}^{\infty}\overline{\psi}(z)w(z,\lambda)dz\sigma_3\right).$$

The desired result follows immediately from this equation. □

As a consequence, we get that the *generating function* $P(\lambda)$ is given by

$$P(\lambda) = \int_{-\infty}^{\infty}\overline{\psi}(x)w(x,\lambda)\mathrm{d}x,$$

and then

$$P(\lambda) = \arccos\left(\frac{1}{2}\operatorname{tr}\mathbf{M}(\lambda)\right). \tag{2.103}$$

To find the relationship between the generating function $P(\lambda)$ and the transition coefficient $a(\lambda)$, we need to use the following formula:

$$|a(\lambda)| = 1 + O(|\lambda|^{-\infty}) \quad \text{as} \quad |\lambda| \to \infty. \tag{2.104}$$

Taking

$$\ln a(\lambda) = \ln|a(\lambda)| + i\arg a(\lambda),$$

Finally, from (2.104) we get

$$\arg a(\lambda) = \frac{1}{i}\ln a(\lambda) + O(|\lambda|^{-\infty}). \tag{2.105}$$

and the following result.

Corollary 2.5.5. *The generating function is equal to the logarithm of the transition coefficient* $a(\lambda)$,

$$\ln a(\lambda) = iP(\lambda) + O(|\lambda|^{-\infty}). \tag{2.106}$$

It follows from Theorem 2.3.1, the asymptotic representation for $\ln a(\lambda)$ is of the form

$$\ln a(\lambda) = i\varkappa \sum_{n=1}^{\infty} \frac{c_n}{\lambda^n} + O(|\lambda|^{-\infty}),$$

where

$$c_n = \int_{-\infty}^{\infty} \overline{\psi}(x) w_n(x,\lambda) dx.$$

We conclude this Section with the analysis of the linear problem in $\mathfrak{sl}(2,\mathbb{C})$, corresponding to the theory developed in Chapter 1. In this particular case, it is possible obtain some results related to the zero curvature equation and time evolution of the invariants. First, we study the linear problem from the point of view of spectral theory and then we apply this approach to derive the final results.

2.5.1 The Spectral Problem.

Consider the linear problem in $\mathfrak{sl}(2,\mathbb{C})$

$$\frac{df}{dx} = \mathbf{U}(x,\lambda)\mathrm{f}, \quad \mathrm{f}(x) = (f_1(x), f_2(x)) \in \mathbb{C}^2, \tag{2.107}$$

$$\mathbf{U}(x,\lambda) = \sqrt{\varkappa}\, \overline{\psi}(x)\sigma_+ + \sqrt{\varkappa}\psi(x)\sigma_- + \frac{\lambda}{2i}\sigma_3.$$

Let $\mathfrak{X} = \left(L_2(\mathbb{R}) \otimes \mathbb{C}^2, \langle , \rangle\right)$ be a Hilbert space with inner product

$$\langle \mathrm{f}(x), \mathrm{g}(x) \rangle = \int_{-\infty}^{\infty} \mathrm{f}(x) \cdot \overline{\mathrm{g}}(x) dx. \tag{2.108}$$

Multiplying both sides of (2.107) by $i\sigma_3$, it is easy to see that the linear problem (2.107) is equivalent to the **eigenvalue problem in** $\mathfrak{X}$

$$\mathscr{L}\mathrm{f} = \frac{\lambda}{2}\mathrm{f}, \tag{2.109}$$

for the first order matrix differential operator

$$\mathscr{L} = i\sigma_3 \frac{d}{dx} + i\sqrt{\varkappa}(\psi(x)\sigma_- - \overline{\psi}(x)\sigma_+). \tag{2.110}$$

So, λ is interpreted as a spectral parameter for $\mathscr{L}$. As it is usual in spectral theory, $\mathscr{L}^*$ denotes the adjoint operator of $\mathscr{L}$, i.e.,

$$\langle \mathscr{L}\mathbf{f}, g\rangle = \langle \mathbf{f}, \mathscr{L}^* g\rangle. \tag{2.111}$$

Using properties of $\sigma_\pm$ (2.10), we get

$$\mathscr{L}^* = i\sigma_3 \frac{d}{dx} + i\text{sign}(\varkappa)\sqrt{\varkappa}(\psi(x)\sigma_- - \overline{\psi}(x)\sigma_+). \tag{2.112}$$

It follows from here that

Case $\varkappa \geq 0$. $\mathscr{L}$ is formally self-adjoint, this means $\mathscr{L} = \mathscr{L}^*$ and its eigenvalues are real;

Case $\varkappa < 0$. $\mathscr{L}$ is not self-adjoint.

Analytic behavior of the transition coefficients. Although, in the decreasing case only it is possible to define the monodromy matrix over the whole real line, the transition coefficient $a(\lambda)$ can be analytically continued into some domain of the complex plane. We derive the analytic continuation for $a(\lambda)$ and its analytic properties, beside of other relevant properties of the transition coefficients.

Using the column notation for the Jost solution

$$\mathbf{T}_\pm(x,\lambda) = (\mathbf{T}_\pm^{(1)}(x,\lambda), \mathbf{T}_\pm^{(2)}(x,\lambda)), \tag{2.113}$$

we formulate the following result.

Lemma 2.5.6. *The transition coefficients have the following expressions*

$$a(\lambda) = \det(\mathbf{T}_-^{(1)}(x,\lambda), \mathbf{T}_+^{(2)}(x,\lambda)), \tag{2.114}$$
$$b(\lambda) = \det(\mathbf{T}_+^{(1)}(x,\lambda), \mathbf{T}_-^{(1)}(x,\lambda)). \tag{2.115}$$

Proof. Denote

$$\mathbf{T}_\pm^{(1)} = \begin{pmatrix} \mathbf{T}_\pm^{(1)1} \\ \mathbf{T}_\pm^{(1)2} \end{pmatrix} \quad \text{and} \quad \mathbf{T}_\pm^{(2)} = \begin{pmatrix} \mathbf{T}_\pm^{(2)1} \\ \mathbf{T}_\pm^{(2)2} \end{pmatrix}.$$

With this notation, we can express the monodromy matrix (2.96) as

$$\mathbf{M}(\lambda) = \begin{pmatrix} \mathbf{T}_+^{(2)2} & -\mathbf{T}_+^{(2)1} \\ -\mathbf{T}_+^{(1)2} & \mathbf{T}_+^{(1)1} \end{pmatrix} \begin{pmatrix} \mathbf{T}_-^{(1)1} & \mathbf{T}_-^{(2)1} \\ \mathbf{T}_-^{(1)2} & \mathbf{T}_-^{(2)2} \end{pmatrix}.$$

By Proposition 2.5.2, we have

$$\begin{aligned} a(\lambda) &= \mathbf{T}_-^{(1)1}\mathbf{T}_+^{(2)2} - \mathbf{T}_-^{(1)2}\mathbf{T}_+^{(2)1} = \det(\mathbf{T}_-^{(1)}(x,\lambda), \mathbf{T}_+^{(2)}(x,\lambda)), \\ b(\lambda) &= \mathbf{T}_+^{(1)1}\mathbf{T}_-^{(1)2} - \mathbf{T}_+^{(1)2}\mathbf{T}_-^{(1)1} = \det(\mathbf{T}_+^{(1)}(x,\lambda), \mathbf{T}_-^{(1)}(x,\lambda)). \end{aligned}$$

□

Proposition 2.5.7. *The transition coefficient $a(\lambda)$ has an analytic extension into the upper half plane* $\mathrm{Im}\lambda \geq 0$, *with the asymptotic behavior*

$$a(\lambda) \to 1 \quad \text{as } |\lambda| \to \infty.$$

Proof. By Lemma (2.5.6),

$$a(\lambda) = \det(\mathbf{T}_-^{(1)}(x,\lambda), \mathbf{T}_+^{(2)}(x,\lambda)).$$

Then, $a(\lambda)$ has an analytic extension, since by Proposition 2.5.1, the column vectors $\mathbf{T}_-^{(1)}(x,\lambda)$ and $\mathbf{T}_+^{(2)}(x,\lambda)$ can be analytically extended into the upper half plane. □

Remark 1. *The analytic properties of $\mathbf{T}_\pm^{(1)}(x,\lambda)$ together with (2.115) imply that, $b(\lambda)$ has no analytic continuation in any neighborhood of the real line.*

Corollary 2.5.8. *The complex valued function $a^*(\lambda) := \overline{a}(\overline{\lambda})$ has an analytic continuation into the lower half plane* $\mathrm{Im}\lambda \leq 0$.

Proof. Suppose that

$$a(\lambda) = u(\lambda) + iv(\lambda), \quad \lambda = x + iy.$$

Taking into account that $a(\lambda)$ is analytic into the upper half plane, we have that the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

holds for all $y \geq 0$. Let

$$a^*(\lambda) = \hat{u}(\lambda) + i\hat{v}(\lambda).$$

Then $\hat{u}(x+iy) = u(x-iy)$ and $\hat{v}(x+iy) = -v(x-iy)$.Hence,

$$\frac{\partial \hat{u}}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial \hat{v}}{\partial y},$$

only if $y \leq 0$. Therefore, $a^*(\lambda) := \overline{a}(\overline{\lambda})$ has an analytic continuation into the lower half plane. □

Proposition 2.5.9. *The transition coefficient $a(\lambda)$ has no zeros on the real line.*

Proof. Assuming there is a real λ_0 such that $a(\lambda_0) = 0$, the normalization relation

$$|a(\lambda)|^2 - |b(\lambda)|^2 = 1$$

implies

$$|b(\lambda_0)|^2 = -1.$$

This is clearly a contradiction. □

The behavior of zeroes of $a(\lambda)$ depends essentially of the sign the parameter $\varkappa$ appearing in (2.37).

Proposition 2.5.10. *If $\varkappa$ is positive, the coefficient $a(\lambda)$ has no zeros on its domain of analyticity.*

Proof. Suppose that $a(\lambda_0) = 0$ with $\text{Im}\lambda_0 > 0$. By Lemma 2.5.6, we have

$$a(\lambda) = \det(\mathbf{T}_-^{(1)}(x,\lambda), \mathbf{T}_+^{(2)}(x,\lambda)). \tag{2.116}$$

Then, it follows that the column vectors $\mathbf{T}_-^{(1)}(x,\lambda_0)$ and $\mathbf{T}_+^{(2)}(x,\lambda_0)$ are linearly dependent. Thus, for $\lambda = \lambda_0$, the linear problem (2.107) has a vector column solution decaying exponentially as $|x| \to \infty$. This fact is equivalent to the spectral problem (2.109) which has a non-real eigenvalue λ_0. On other hand the operator $\mathscr{L}$ is self-adjoint for $\varkappa > 0$, and only has real eigenvalues. This contradiction shows that $a(\lambda)$ does not have complex zeros. □

The situation is quit different for $\varkappa < 0$. Since $\mathscr{L}$ is not self-adjoint $a(\lambda)$ may have zeros. The analyticity and the asymptotic behavior of the coefficient $a(\lambda)$ (Proposition 2.5.7) imply that the zeros are located in a bounded region of the half plane $\text{Im}\lambda \geq 0$ and may only accumulate towards the real line.

In order to simplify our analysis, we make the following assumption:

- *all the zeroes are simple.*

Proposition 2.5.11. *Under these assumptions, $a(\lambda)$ has only a finite number of zeros and there is a strict inequality for $|b(\lambda)| < 1$ for $|\lambda| \to \infty$.*

This result follows directly from the normalization relation (2.68)

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the zeros of $a(\lambda)$, with $\text{Im}\lambda_j > 0$, $j = 1, 2, ..., n$. From the Lemma (2.5.6), we have that the column of $\mathbf{T}_-^{(1)}(x,\lambda)$ is proportional to $\mathbf{T}_+^{(2)}(x,\lambda)$. Then, for each j, there is a complex number γ_j such that

$$\mathbf{T}_-^{(1)}(x,\lambda_j) = \gamma_j \mathbf{T}_+^{(2)}(x,\lambda_j), \quad j = 1, 2, ..., n. \tag{2.117}$$

We remark that

$$\gamma_j \neq 0, \tag{2.118}$$

because if this do not occur the monodromy matrix has determinant equal to zero.

Using the involution property, we find that

$$\mathbf{T}_-^{(1)}(x, \overline{\lambda}_j) = \overline{\gamma}_j \mathbf{T}_+^{(2)}(x, \overline{\lambda}_j).$$

In summary, the spectral problem (2.109) has the following characteristics:

i) For any $\varkappa$, $\mathscr{L}$ has continuous spectrum of multiplicity two on the whole real line, according with the existence of two linearly independent solution of the linear problem (2.107).

ii) If $\varkappa > 0$, there are no more solutions of the spectral problem (2.109).

iii) If $\varkappa < 0$, there exist a finite set of number in the upper complex plane, which are solution of the spectral problem (2.109). This set constitutes the discrete spectrum.

According to observations below, we shall call $a(\lambda)$ and $b(\lambda)$ the transitions coefficients for the **continuous spectrum**. The constants γ_j,$\overline{\gamma}_j$, and the complex numbers λ_j, $j = 1, 2, ..., n$ will be called the transition coefficients for the **discrete spectrum**.

Definition 2.5.1. *The set $\{b(\lambda), \lambda_j, \gamma_j\}$ is called the spectral data of the linear problem (2.39) if the following condition hold*

- $\mathrm{Im}\lambda_j > 0$,
- λ_j *are isolated zeroes, and*
- $\gamma_j \neq 0$.

It is possible to express the function $a(\lambda)$ in terms of $b(\lambda)$ and its zeroes on the complex plane. Using some facts from the theory of analytic functions [1, 13], we give this results.

Lemma 2.5.12. *Suppose we are given a complex-valued function $g(\lambda)$ which is*

- *analytic for* $\mathrm{Im}\lambda > 0$,
- *continuos for* $\mathrm{Im}\lambda \geq 0$,
- *vanishing at the infinity*

$$\lim_{|\lambda| \to \infty} g(\lambda) = 0. \tag{2.119}$$

Then, the real $\mathrm{Re}g(\lambda)$ *and imaginary* $\mathrm{Im}g(\lambda)$ *parts are related by the formula*

$$\mathrm{Im}g(\lambda) = -\frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{Re}g(\lambda)}{\mu-\lambda}\mathrm{d}\mu, \quad \lambda \in \mathbb{R}. \tag{2.120}$$

Here

$$\mathrm{p.v.}\int_{-\infty}^{\infty}\frac{g(\mu)}{\mu-z}\mathrm{d}\mu = \lim_{\epsilon\to 0}\left(\int_{-\infty}^{z-\epsilon}\frac{g(\mu)}{\mu-z}\mathrm{d}\mu + \int_{z+\epsilon}^{\infty}\frac{g(\mu)}{\mu-z}\mathrm{d}\mu\right) \tag{2.121}$$

is the principal value integral.

Proof. By the Cauchy formula we have

$$g(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(\mu)}{\mu-\lambda}\mathrm{d}\mu, \ \mathrm{Im}\lambda > 0. \tag{2.122}$$

The Sokhotskii-Plemlj formula [1, 13] implies that

$$\begin{aligned} g(z+0) &\stackrel{\mathrm{def}}{=} \lim_{\lambda\to z}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(\mu)}{\mu-\lambda}\mathrm{d}\mu \\ &= \frac{g(z)}{2} + \frac{1}{2\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty}\frac{g(\mu)}{\mu-z}\mathrm{d}\mu \end{aligned} \tag{2.123}$$

for every $z \in \mathbb{R}$. On other hand, since $g(\lambda)$ is continuous, we have

$$g(z) = g(z+0) = \frac{g(z)}{2} + \frac{1}{2\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty}\frac{g(\mu)}{\mu-z}\mathrm{d}\mu. \tag{2.124}$$

This implies

$$g(z) = \frac{1}{\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty}\frac{g(\mu)}{\mu-z}\mathrm{d}\mu \tag{2.125}$$

□

Now, we apply this Lemma to the function $g(\lambda) = \ln a(\lambda)$. Then,

$$g(\lambda) = \ln|a(\lambda)| + i\mathrm{Arg}a(\lambda).$$

By the normalization relation (2.68), we have

$$|a(\lambda)|^2 = 1 + \varepsilon|b(\lambda)|^2, \quad \varepsilon = \mathrm{sign}\varkappa. \tag{2.126}$$

So, we have two cases. **Case** $\varkappa < 0$. Since the function $a(\lambda)$ is analytic in the upper half plane, continuous for $\mathrm{Im}\lambda \geq 0$ and has the asymptotic behavior

$$a(\lambda) = 1 + O(1) \quad \text{as} \quad |\lambda| \to \infty, \tag{2.127}$$

We let denote $\lambda_1, ..., \lambda_n$ the zeroes of $a(\lambda)$ in the upper half plane. Introducing the function

$$\widetilde{a}(\lambda) = \prod_{i=1}^{n} \frac{\lambda - \overline{\lambda}_i}{\lambda - \lambda_i} \cdot a(\lambda), \tag{2.128}$$

which is also analytic, $\widetilde{a}(\lambda) \neq 0$ for $\mathrm{Im}\lambda > 0$, and satisfies

$$|\widetilde{a}(\lambda)|^2 = |a(\lambda)|^2 = 1 - |b(\lambda)|^2, \quad \lambda \in \mathbb{R}. \tag{2.129}$$

we have that the function

$$g(\lambda) \stackrel{\mathrm{def}}{=} \ln \widetilde{a}(\lambda) = \ln |\widetilde{a}(\lambda)| + i\arg\widetilde{a}(\lambda). \tag{2.130}$$

satisfies the hypotheses of Lemma 2.5.12. Taking into account that

$$\mathrm{Re}g(\lambda) = \ln |\widetilde{a}(\lambda)| = \frac{1}{2} \ln(1 - |b(\lambda)|^2), \tag{2.131}$$

we deduce from (2.120) the following representation

$$\mathrm{Im}g(\lambda) = \arg\widetilde{a}(\lambda) = -\frac{1}{2\pi}\mathrm{p.v.} \int_{-\infty}^{\infty} \frac{\ln(1 - |b(\lambda)|^2)}{\mu - \lambda} \mathrm{d}\mu, \tag{2.132}$$

for $\lambda \in \mathbb{R}$. So, we have proved the following result.

Theorem 2.5.13. *For every $\lambda \in \mathbb{R}$, we have*

$$a(\lambda) = \left(\prod_{i=1}^{n} \frac{\lambda - \overline{\lambda}_i}{\lambda - \lambda_i} \right) \cdot (1 - |b(\lambda)|^2)^{\frac{1}{2}} \cdot \exp\left(\frac{1}{2\pi i} \mathrm{p.v.} \int_{-\infty}^{\infty} \frac{\ln(1 - |b(\lambda)|^2)}{\mu - \lambda} \mathrm{d}\mu \right) \tag{2.133}$$

One can rewrite (2.133), using the Sokhotski type formula:

$$\mathrm{p.v.} \int_{-\infty}^{\infty} \frac{f(\mu)}{\mu - \lambda} \mathrm{d}\mu = \pi i f(\mu) - \int_{-\infty}^{\infty} \frac{f(\mu)}{\mu - \lambda - 0i} \mathrm{d}\mu. \tag{2.134}$$

Here $f(\mu) \to 0$ as $|\mu| \to \infty$ and

$$\int_{-\infty}^{\infty} \frac{f(\mu)}{\mu - \lambda - 0i} \mathrm{d}\mu = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{f(\mu)}{\mu - \lambda - \epsilon i} \mathrm{d}\mu. \tag{2.135}$$

Applying (2.123) for $f(\mu) = \ln(1 - |b(\lambda)|^2)$, we get

$$a(\lambda) = \prod_{i=1}^{n} \left(\frac{\lambda - \overline{\lambda}_i}{\lambda - \lambda_i} \right) \cdot \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 - |b(\lambda)|^2)}{\mu - \lambda - 0i} \mathrm{d}\mu \right). \tag{2.136}$$

One can show that this formula is valid for $\mathrm{Im}\lambda \geq 0$.

Case $\varkappa \geq 0$. In this case, $a(\lambda)$ is analytic in the upper half plane and has no zeroes. Applying the same arguments as in the previous case to $g(\lambda) = \ln a(\lambda)$, we get

$$a(\lambda) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 + |b(\lambda)|^2)}{\mu - \lambda} \mathrm{d}\mu \right). \tag{2.137}$$

The Time Evolution of Spectral Data

Recall that the evolution equation (see Section 1.4), for the fundamental solution is of the form

$$\frac{\partial}{\partial t}\mathbf{T}(x,y,\lambda) = \mathbf{V}(x,\lambda)\mathbf{T}(x,y,\lambda) - \mathbf{T}(x,y,\lambda)\mathbf{V}(y,\lambda).$$

Here

$$\mathbf{V}(x,\lambda) = \mathbf{V}_0(x,\lambda) + \frac{i\lambda^2}{2}\sigma_3, \tag{2.138}$$

where $\mathbf{V}_0(x,\lambda) \in L_1^{2\times 2}(\mathbb{R})$ and λ is a real parameter. Taking limits in (2.138) as $x \to \infty$, $y \to \pm\infty$ and multiplying the right hand side by $\mathbf{E}(y,\lambda)$, we get

$$\mathbf{V}(x,\lambda) \to \mathbf{V}(\lambda) = \frac{i\lambda^2}{2}\sigma_3, \quad \text{as} \quad |x| \to \infty.$$

Therefore, we obtain

$$\frac{\partial}{\partial t}\mathbf{T}_\pm(x,\lambda) = \mathbf{V}(x,\lambda)\mathbf{T}_\pm(x,\lambda) - \frac{i\lambda^2}{2}\mathbf{T}_\pm(x,\lambda)\sigma_3.$$

Performing the same operation with respect to x, we obtain the equation for the monodromy matrix

$$\frac{\partial}{\partial t}\mathbf{M}(\lambda,t) = \frac{i\lambda^2}{2}\left[\sigma_3, \mathbf{M}(\lambda,t)\right].$$

This equation posses a remarkable property: the dependence on $\psi(x)$, $\overline{\psi}(x)$ is completely eliminated. In terms of the transition coefficients for the continuous spectrum, the last equation is equivalent to

$$\frac{\partial}{\partial t}a(\lambda,t) = 0 \tag{2.139}$$

$$\frac{\partial}{\partial t}b(\lambda,t) = -i\lambda^2 b(\lambda,t). \tag{2.140}$$

In particular, we deduce that $a(\lambda,t)$ is time independent

$$a(\lambda,t) = a(\lambda,0), \quad \forall \in \mathbb{R}.$$

By the analyticity property, the same holds for $\mathrm{Im}\lambda > 0$, so that the zeros λ_j are time independent as well. Thus, in the decreasing case, the generating function for the conservation laws is just equal to $a(\lambda)$.

Finally, let us describe the evolution of the transition coefficients for the discrete spectrum. We have $\mathbf{T}_-^{(1)}(x,\lambda)$ and $\mathbf{T}_+^{(2)}(x,\lambda)$,

$$\frac{\partial}{\partial t}\mathbf{T}_{-}^{(1)}(x,\lambda) = \mathbf{V}(x,\lambda)\mathbf{T}_{-}^{(1)}(x,\lambda) - \frac{i\lambda^2}{2}\mathbf{T}_{-}^{(1)}(x,\lambda), \tag{2.141}$$

$$\frac{\partial}{\partial t}\mathbf{T}_{+}^{(2)}(x,\lambda) = \mathbf{V}(x,\lambda)\mathbf{T}_{+}^{(2)}(x,\lambda) - \frac{i\lambda^2}{2}\mathbf{T}_{+}^{(2)}(x,\lambda). \tag{2.142}$$

These relations also hold for $\mathrm{Im}\lambda > 0$. They are compatible with

$$\mathbf{T}_{-}^{(1)}(x,\lambda_j) = \gamma_j \mathbf{T}_{+}^{2}(x,\lambda_j), \quad j = 1,2,...,n, \tag{2.143}$$

only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_j(t) = -i\lambda_j^2\gamma_j(t). \tag{2.144}$$

The last equation and the differential equation for $a(\lambda)$ and $b(\lambda)$ can easily be solved so that the time dependence of transition coefficients is given by the simple formulae

$$b(\lambda,t) = \mathrm{e}^{-i\lambda^2 t}b(\lambda,0), \tag{2.145}$$

$$\gamma_j(t) = \mathrm{e}^{-i\lambda^2 t}\gamma_j(0). \quad j = 1,2,...,n. \tag{2.146}$$

Chapter 3

The Inverse Problem. The Rapidly Decreasing Case.

First, we give some explanations about what the inverse problem means. In Chapter 2, we applied the general results of Chapter 1 and we derived the monodromy matrix for the linear system in $\mathfrak{sl}(2,\mathbb{C})$, supplied with two boundary conditions: quasi-periodic case and decreasing cases. The main characteristic in those particular cases is the special form of the monodromy matrix which arise the transition coefficients. After that, we developed the analytic properties of the transition coefficient, and the time evolution of the spectral data related to zero curvature equation. The inverse problem consisting of it is possible to reconstruct the linear system in $\mathfrak{sl}(2,\mathbb{C})$ from its spectral data (definition 2.5.1). Moreover, If it is possible to use a linear time dynamics for the spectral data (2.145) and to get some solutions for the zero curvature equation (1.33). We show that it is possible to solve the inverse problem, in the rapidly decreasing. This is a special case of the linear problems studied in Chaprter 2.

3.1 Formulation of main Results

Recall that the Schwartz space $\mathcal{S}(\mathbb{R})$ consists of all functions $f \in C^{\infty}(\mathbb{R})$ such that for each integer $n \geq 0$

$$|x^m| \left|\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right| \to 0 \quad \text{as} \quad |x| \to \infty, \tag{3.1}$$

for any $m > 0$. It is clear that for every $f \in \mathcal{S}(\mathbb{R})$, we have $\frac{\mathrm{d}^n f}{\mathrm{d}x^n} \in L_1(\mathbb{R})$. Given a $\psi \in \mathcal{S}(\mathbb{R})$, we consider the follow linear problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x,\lambda) = \left(\mathrm{U}_0(x) + \frac{\lambda}{2i}\sigma_3\right)\mathrm{f}(x,\lambda), \tag{3.2}$$

$$\tag{3.3}$$

where

$$\mathrm{U}_0(x) = \sqrt{\varkappa}(\psi(x)\sigma_+ + \overline{\psi}(x)). \tag{3.4}$$

In this case, we say that (3.2) is *the linear problem associated to ψ with the rapidly decreasing coefficients.*

The form of the coefficient $\mathbf{U}(x,\lambda)$ in (3.2) implies that involution property holds (Proposition 2.1.1) and hence, its monodromy matrix $\mathbf{M}(\lambda)$ (2.96) can be written as

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & \epsilon\overline{b}(\overline{\lambda}) \\ b(\lambda) & \overline{a}(\overline{\lambda}) \end{pmatrix},$$

where $\epsilon = \mathrm{sign}(\varkappa)$ (Proposition 2.1.3). In Section 2.2 , we saw that in the decreasing case the linear system (3.2) is interpreted as the spectral problem:

$$\mathscr{L}\mathrm{f} = \frac{\lambda}{2}\mathrm{f}, \tag{3.5}$$

where $\mathscr{L}$ is the differential linear operator given by (2.109). This gives rise to the notion of spectral data for the linear problem (3.2),(??) which consist of the function $b(\lambda)$ for $\varkappa > 0$ (when $\mathscr{L}$ is self-adjoint), and the set $\{(b(\lambda), \overline{b}(\lambda), \lambda_j, \overline{\lambda}_j, \gamma_j, \overline{\gamma_j})\}$ for $\varkappa < 0$ (when $\mathscr{L}$ is not self-adjoint).

Follow [8], by inverse problem we mean the reconstruction of the matrix coefficients of $\mathbf{U}(x,\lambda)$ in (3.2) from spectral data. Actually, we only need to reconstruct the complex valued $\psi(x)$. We have two main results for the inverse problem, each one of them depends on wether the linear operator $\mathscr{L}$ (3.5) is self-adjoint or not.

Theorem 3.1.1 ($\mathscr{L}$ is self-adjoint). *Let $b(\lambda) \in \mathcal{S}(\mathbb{R})$ be a complex valued function satisfying*

$$|b(\lambda)| < 1. \tag{3.6}$$

Let

$$a(\lambda) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(1+|b(\mu)|)}{\mu-\lambda-i0}\mathrm{d}\mu\right\}. \tag{3.7}$$

Then, there exist a unique $\psi(x) \in \mathcal{S}(\mathbb{R})$ such that

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & \overline{b}(\lambda) \\ b(\lambda) & \overline{a}(\lambda) \end{pmatrix}$$

is the monodromy matrix of the linear problem (3.2) associated to $\psi(x)$*. Moreover (3.6) and (3.7 implies)*

$$|a(\lambda)|^2 - |b(\lambda)|^2 = 1, \quad \textit{for all real } \lambda. \tag{3.8}$$

Theorem 3.1.2 ($\mathscr{L}$ is not self-adjoint). *Let* $\{(b(\lambda), \bar{b}(\lambda), \lambda_j, \overline{\lambda}_j, \gamma_j, \overline{\gamma_j})\}$ *be a set consisting of complex number* λ_j, γ_j $(j = 1, 2, ..., n)$ *and* $b(\lambda) \in \mathcal{S}(\mathbb{R})$ *which satisfy the following conditions:*

- $\mathrm{Im}\lambda_j > 0$, $\lambda_i \neq \lambda_j$ *if* $i \neq j$,
- $\gamma_j \neq 0$,
- $|b(\lambda)| < 1$.

Let

$$a(\lambda) = \prod_{j=1}^{n} \frac{\lambda - \lambda_j}{\lambda - \overline{\lambda}_j} \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln(1 - |b(\mu)|)}{\mu - \lambda - i0} \mathrm{d}\mu\right\}. \tag{3.9}$$

Then there exist an unique $\psi(x) \in \mathcal{S}(\mathbb{R})$ *such that*

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & -\bar{b}(\overline{\lambda}) \\ b(\lambda) & \bar{a}(\overline{\lambda}) \end{pmatrix}$$

is the monodromy matrix of the linear problem (3.2) associated to $\psi(x)$*. Moreover,*

1. $|a(\lambda)|^2 + |b(\lambda)|^2 = 1$*, for all real* λ*;*
2. *the columns* $\left(\mathbf{T}_-^{(1)}(x,\lambda), \mathbf{T}_+^{(2)}(x,\lambda)\right)$ *of the Jost solutions* $\mathbf{T}_\pm(x,\lambda)$ *in (2.91)-(2.92) satisfy the conditions*

$$\mathbf{T}_-^{(1)}(x,\lambda_j) = \gamma_j \mathbf{T}_+^{(2)}(x,\lambda_j), \quad j = 1, 2, ..., n. \tag{3.10}$$

Remark 2. *The inverse problem can be formulated in terms of spectral problem and spectral data:*

1. *For a given function* $b(\lambda)$ *in the Schwartz space* $\mathcal{S}(\mathbb{R}_\lambda)$ *satisfying the conditions of Theorem 3.1.1, there exists a unique complex valued function* $\psi \in \mathcal{S}(\mathbb{R})$ *such that the complex functions* $a(\lambda)$ *(3.7) and* $b(\lambda)$ *are the transition coefficients of the continuous spectrum of linear operator*

$$\mathscr{L} = i\sigma_3\frac{d}{dx} + i\sqrt{\varkappa}(\psi(x)\sigma_- - \overline{\psi}(x)\sigma_+),$$

where $\mathscr{L}$ *is self-adjoint , and* $\varkappa$ *is a positive real.*

2. *For a given set* $\{(b(\lambda), \overline{b}(\lambda), \lambda_j, \overline{\lambda}_j, \gamma_j, \overline{\gamma_j})\}$ *satisfying the conditions of Theorem (3.1.2), there exists a unique complex valued function* $\psi \in \mathcal{S}(\mathbb{R})$ *such that the complex functions* $a(\lambda)$ *(3.9),* $b(\lambda)$ *and the complex number* λ_i*,* γ_i *constitute the transition coefficients of the continuous and discrete spectrum of linear operator*

$$\mathscr{L} = i\sigma_3 \frac{d}{dx} - i\sqrt{|\varkappa|}(\psi(x)\sigma_- - \overline{\psi}(x)\sigma_+);$$

where $\mathscr{L}$ *is not self-adjoint and* $\varkappa$ *is a negative real number.*

The basic tool for solving the inverse problem formulated in the preceding Theorems is provided by the **Riemann-Hilbert Problem** or the analytic factorization problem [8].

3.2 The Riemann-Hilbert Problem

In [1], a complete chapter is devoted to the **Riemann-Hilbert Problem**. The authors present distinct formulations of Riemann Problems and its applications. Here, we describe the **Matrix Riemann-Hilbert Problem**, and we introduce it in an adequate form for our purposes. A better treatment of the Riemann-Hilbert Problems can be found in the mentioned reference.

3.2.1 The Scalar Riemann Problem on the Real Axis

In many applications one encounters Riemann-Hilbert problems formulated on the real axis, [1]. This problem is well defined in the space of integrable function. We shall consider in more detail, the set Λ_1 consisting of the function $F(x)$ of the form

$$F(\lambda) = c + \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\lambda x} dx, \quad c \in \mathbb{C}, \tag{3.11}$$

where $f(\lambda)$ is continuous function and $f \in \mathrm{L}_1(\mathbb{R}_\lambda)$. Let Λ_1^- be the subset of Λ_1 consisting of functions of the form

$$F_-(\lambda) = c + \int_0^{\infty} f(x)\mathrm{e}^{-i\lambda x} dx. \tag{3.12}$$

Putting $\lambda = \alpha + i\beta$, it is clear that every function in Λ_1^- is analytically continuable in the lower half plane, [1, 13]. Similarly, Λ_1^+ consists of the functions of the form

$$F_+(\lambda) = c + \int_{-\infty}^{0} f(x)\mathrm{e}^{-i\lambda x} dx, \tag{3.13}$$

which are analytically continuable in the upper half plane.

Formulation of the Riemann Problem

Let $f(x)$ and $g(x)$ be continuous and absolutely integrable functions on $\mathbb{R}$. The problem is to search a function $\varphi(x)$ whose Fourier transform belongs to Λ_1 and satisfies the following integral equation

$$\varphi(x) - \int_0^\infty g(x-t)\varphi(t)dt = f(x), \qquad x > 0. \tag{3.14}$$

Indeed, the Riemann problem formulated in equation (3.14) is equivalent to a factorization problem, which is the usual setting of the Riemann Problems. We derive this relationship.

First, we extend the equation (3.14) to all the real axis. Let

$$\widetilde{\varphi}(x) = \begin{cases} 0 & x < 0 \\ \varphi(x) & x > 0 \end{cases}. \tag{3.15}$$

Then, equation (3.14) is rewritten as

$$\widetilde{\varphi}(x) - \int_{-\infty}^\infty g(x-t)\widetilde{\varphi}(t)dt = f(x), \qquad x > 0, \tag{3.16}$$

and

$$\widetilde{\varphi}(x) - \int_{-\infty}^\infty g(x-t)\widetilde{\varphi}(t)dt = -\int_0^\infty g(x-t)\varphi(t)dt = h(x), \qquad x < 0, \tag{3.17}$$

where $h(x)$ is an unknown function. Let

$$\widetilde{f}(x) = \begin{cases} 0 & x < 0 \\ f(x) & x > 0 \end{cases}, \tag{3.18}$$

and

$$\widetilde{h} = \begin{cases} h(x) & x < 0 \\ 0 & x > 0 \end{cases}. \tag{3.19}$$

Equations (3.16) and (3.17) are equivalent to

$$\widetilde{\varphi}(x) - \int_{-\infty}^\infty g(x-t)\widetilde{\varphi}(t)dt = \widetilde{f}(x) + \widetilde{h}(x), \qquad -\infty < x < \infty. \tag{3.20}$$

Denoting by Φ, F, H, G the Fourier transform of $\widetilde{\varphi}$, $\widetilde{f}$, $\widetilde{h}$, g, respectively, and applying the Fourier transform of the above equation we get

$$(1 - G(\lambda))\Phi(\lambda) = F(\lambda) + H(\lambda). \tag{3.21}$$

Since $\widetilde{\varphi} = 0$ when $x < 0$, we have

$$\Phi(\lambda) = \int_0^\infty \widetilde{\varphi}(x) e^{-i\lambda x} dx, \quad i.e.\ \Phi(\lambda) \in \Lambda_1^-. \tag{3.22}$$

Analogously, $F(\lambda) \in \Lambda_1^-$ and $H(\lambda) \in \Lambda_1^+$. Therefore, the Riemann problem (3.14) defines the Factorization Problem

$$H^+(\lambda) = (1 - G(\lambda))\Phi^-(\lambda) - F^-(\lambda), \tag{3.23}$$

where F^- and $G($ are known (because $f(x)$ and $g(x)$ are given) and the unknowns are H^+ and Φ^-. The solvability of the Riemann Problem (3.23) is established in the following results:

Theorem 3.2.1. *Let $f(x)$ and $g(x)$ be continuous and absolutely integrable functions such that their Fourier transform denoted by $F(\lambda)$, $G(\lambda)$ satisfy*

(i) Hölder condition on $\mathbb{R}$, i.e. for any two points λ_1 and λ_2 on $\mathbb{R}$

$$|G(\lambda_1) - G(\lambda_2)| \leq L|\lambda_1 - \lambda_2|^\gamma, \quad L > 0, \quad 0 < \gamma \leq 1.$$

(ii) $1 - G(\lambda) \neq 0$ for $\lambda \in \mathbb{R}$,

(iii) The index of $(1 - G(\lambda))$, given by $\mathrm{ind}[1 - G(\lambda)] = \frac{1}{2\pi}[\arg(1-G)]_{-\infty}^{\infty}$, is zero.

Then, there exists an unique solution of the Riemann (3.23) vanishing at infinite.

Proposition 3.2.2. *The solution of the homogeneous factorization problem*

$$\Phi^+(\lambda) = (1 - G(\lambda))\Phi^-(\lambda), \quad -\infty < \lambda < \infty \tag{3.24}$$

with the asymptotic conditions

$$\Phi^\pm(\lambda) \to 1, \quad as \quad |\lambda| \to \infty; \tag{3.25}$$

is given by

$$X(\lambda) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln((1 - G(\mu)))}{\mu - \lambda} d\mu, \right] \quad \lambda \in \mathbb{C}. \tag{3.26}$$

We omit the proofs of these results because they are out of our purposes. A detail development of these proofs can be found in [1].

The solutions $\Phi^\pm$ of the factorization problem (3.23), taking $\Phi^+ = H^+$, are given by

$$\Phi^+(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Phi(\mu)}{\mu - \lambda} d\mu, \quad \mathrm{Im}\lambda > 0, \tag{3.27}$$

$$\Phi^-(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Phi(\mu)}{\mu - \lambda} d\mu, \quad \mathrm{Im}\lambda < 0; \tag{3.28}$$

where

$$\Phi(\lambda) = \frac{-X(\lambda)}{2\pi i}\int_{-\infty}^{\infty}\frac{F^{-}(\mu)}{X^{+}(\mu)(\mu-\lambda)}d\mu, \tag{3.29}$$

and $X(\lambda)$ is the solution of homogeneous problem (3.24). Finally, the solution $\varphi(x)$ of the Riemann Problem (3.14) is obtained by taking the inverse Fourier transform of $\Phi^{-}(\lambda)$.

3.2.2 The Matrix Riemann Problems on the real axis

According to [1], the matrix Riemann problem, in general, is more complicated than the scalar problem. Generally, the solution can not be expressed in closed form; in fact, it is characterized through a system of linear integral equations.

Here, we consider the matrix homogeneous Riemann problem on the real axis. In [10], Gohberg and Krein (1958) led their investigation of certain matrix integral equation, which conducted to study the factorization problem

$$\mathbf{G}(\lambda) = \mathbf{G}^{+}(\lambda)\mathbf{G}^{-}(\lambda), \qquad \lambda \text{ on } \mathbb{R}, \tag{3.30}$$

and

$$\mathbf{G}^{\pm}(\lambda) \to \mathbf{I}_n \quad \text{as} \quad |\lambda| \to \infty; \tag{3.31}$$

where $\mathbf{G}$ is an $n \times n$ nonsingular matrix whose components belong to the set Λ_1 defined in equation (3.11), and $\mathbf{G}^{+}(\lambda)$ and $\mathbf{G}^{-}(\lambda)$ are analytic in the upper and lower half plane (actually, the components of $\mathbf{G}^{\pm}$ belong to $\Lambda_1^{\pm}$).

The factorization problem (3.30) is closely related with the following homogeneous Riemann problem: given an $n \times n$ matrix $\mathbf{G}(\lambda)$ that satisfies a Hölder condition and is nonsingular (i.e. all the matrix elements $\{G_{ij}\}$ satisfy a Hölder condition and $\det \mathbf{G}(\lambda) \neq 0$ on $\mathbb{R}$), find a matrix function $\mathbf{\Phi}(\lambda)$, such that

$$\mathbf{\Phi}^{+}(\lambda) = \mathbf{G}(\lambda)\mathbf{\Phi}^{-}(\lambda), \qquad \lambda \text{ on } \mathbb{R}, \tag{3.32}$$

where

- $\mathbf{\Phi}^{+}(\lambda)$ is a continuous function for $\mathrm{Im}\lambda \geq 0$ and analytic on the upper half complex plane,
- $\mathbf{\Phi}^{+}(\lambda)$ is a continuous function for $\mathrm{Im}\lambda \leq 0$ and analytic on the lower half complex plane, and
- $\mathbf{\Phi}^{\pm}(\lambda) \to \mathbf{\Phi}(\lambda)$ as $\mathrm{Im}\lambda \to 0$.

The following result was proven by Gohberg and Krein in [10].

Proposition 3.2.3. *Let $\mathbf{G}(\lambda)$ be a complex matrix. Assuming that the real or imaginary part of is definite, $\{G(\lambda)\}_{ij} \in \Lambda_1$ and $\det \mathbf{G}(\lambda) \neq 0$; then all its individual indices are zero.*

The real and imaginary parts of the matrix $\mathbf{G} = \mathbf{G}_R + i\mathbf{G}_I$ are defined by

$$\mathbf{G}_R \equiv \frac{1}{2}\left(\mathbf{G} + \mathbf{G}^*\right), \quad \mathbf{G}_I \equiv \frac{1}{2}\left(\mathbf{G} - \mathbf{G}^*\right), \quad \mathbf{G}^* = \overline{\mathbf{G}^T}; \tag{3.33}$$

where the superscript T denotes transpose and the over bar denotes complex conjugation. The unique solvability of the factorization problem (3.30) is establish in the following Theorem due to Gohberg and Krein.

Theorem 3.2.4. *If $\mathbf{G}(\lambda) + \mathbf{G}(\lambda)^*$ is positive definite then the factorization problem (3.30) has a unique pair of solutions $\mathbf{G}^+(\lambda)$ and $\mathbf{G}^+(\lambda)$ whose components belong $\Lambda_1^\pm$, are non-degenerates on their domains of analyticity and $\mathbf{G}^\pm(\lambda) \to \mathbf{I}_n$ as $|\lambda| \to \infty$.*

We do not give the prove of the below results, which can be found in [1, 8, 10].

3.3 The Proof of Main Results

The Theorems formulated in the Section 3.1 establish formally the existence of a linear problem for the given spectral data. However, they do not say how the linear problem can be obtained. After we prove the Theorems 3.1.1 and 3.1.2, we describe how to reconstruct the linear problem. There is no difference in the proof of both theorems up in the application of the Riemann Problem.

We give the proof in several steps.

Step 1. We let assume that the complex valued function $b(\lambda)$ satisfy the conditions of wether Theorem 3.1.1 or Theorem 3.1.2. We define the matrix valued function $\mathbb{R}^2 \ni (x, y) \mapsto \mathbf{G}(x, \lambda)$

$$\begin{aligned} \mathbf{G}(x,\lambda) &\stackrel{\text{def}}{=} \mathbf{E}(x,\lambda)\mathbf{G}(\lambda)\mathbf{E}^{-1}(x,\lambda) \\ &= \begin{pmatrix} a(\lambda)\mathrm{e}^{-i\lambda x} & \epsilon\bar{b}(\lambda)\mathrm{e}^{-i\lambda x} \\ b(\lambda)\mathrm{e}^{i\lambda x} & \overline{a}(\lambda)\mathrm{e}^{i\lambda x} \end{pmatrix}, \end{aligned}$$

with

$$\mathbf{E}(x,\lambda) = \exp\left\{\frac{x\lambda}{2i}\sigma_3\right\}, \quad \mathbf{G}(\lambda) = \begin{pmatrix} a(\lambda) & \epsilon\bar{b}(\lambda) \\ b(\lambda) & \overline{a}(\lambda) \end{pmatrix}, \tag{3.34}$$

and $\varepsilon = 1$ or -1, depending if $b(\lambda)$ satisfies the condition of Theorem 3.1.1 or 3.1.2.

The matrix $\mathbf{G}(x, \lambda)$ has the following properties:

1. $\det \mathbf{G}(x,\lambda) = 1 + \epsilon|b(\lambda)|^2 > 0$, because $|b(\lambda)| < 1$.
2. Since $b(\lambda) \in \mathcal{S}(\mathbb{R}_\lambda)$ we have $\mathbf{G}(x,\lambda) = \mathbf{I} + o(1)$, as $|\lambda| \to \infty$
3. The components of $\mathbf{G}(x,\lambda)$ belong to λ_1, then $\mathbf{G}(x,\lambda)$ has the integral representation

$$\mathbf{G}(x,\lambda) = \mathbf{I} + \int_{-\infty}^{\infty} \Phi(x+s)\mathrm{e}^{i\lambda s}\mathrm{d}s, \tag{3.35}$$

where

$$\Phi(s) = \begin{pmatrix} 0 & \epsilon\overline{\beta}(-s) \\ -\beta(s) & 0 \end{pmatrix}, \tag{3.36}$$

and $\beta(s)$ is the Fourier transform of $b(\lambda)$,

$$\beta(s) = \mathcal{F}\{b(\lambda)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(\lambda)\mathrm{e}^{-i\lambda s}\mathrm{d}\lambda. \tag{3.37}$$

It is easily to realize that $\mathbf{G}(x,\lambda) + \mathbf{G}(x,\lambda)^*$ is positive definite. For $\epsilon = 1$,

$$\frac{1}{2}(\mathbf{G}(x,\lambda) + \mathbf{G}(x,\lambda)^*) = \mathbf{I}, \tag{3.38}$$

and for $\epsilon = -1$, $\mathbf{G}(x,\lambda)$ is Hermitian positive definite which has no zeroes on the real axis.

By Theorem 3.2.4, there exit unique matrices $\mathbf{G}_\pm(x,\lambda)$ such that

- $\mathbf{G}_+(x,\lambda)$ is analytic into the upper half plane,i.e., if $\mathrm{Im}\lambda \geq 0$.
- $\mathbf{G}_-(x,\lambda)$ is analytic into the lower half plane, i.e., if $\mathrm{Im}\lambda \leq 0$.
- The components of $\mathbf{G}^+(x,\lambda)$ belong to $\lambda_1^\pm$.
- For each $\lambda \in (-\infty,\infty)$,

$$\mathbf{G}(x,\lambda) = \mathbf{G}_+(x,\lambda)\mathbf{G}_-(x,\lambda). \tag{3.39}$$

Step 2. We let define the matrices

$$\mathbf{T}_+(x,\lambda) = \mathbf{G}_+^{-1}(x,\lambda)\mathbf{E}(x,\lambda) \tag{3.40}$$

and

$$\mathbf{T}_-(x,\lambda) = \mathbf{G}_-(x,\lambda)\mathbf{E}(x,\lambda). \tag{3.41}$$

The matrices $\mathbf{T}_\pm$ posses the following properties

- $\mathbf{T}_+$ is analytic and non-degenerate in the upper half plane, except, maybe, for simple poles at $\lambda = \lambda_j$, $j = 1, 2, .., n$.

- $\mathbf{T}_-$ is analytic and non-degenerate in the lower half plane, except, maybe, for simple poles at $\lambda = \overline{\lambda}_j$, $j = 1, 2, .., n$.

- $\mathbf{T}_\pm$ satisfy

$$\mathbf{T}_-(x, \lambda) = \mathbf{T}_+(x, \lambda)\mathbf{G}(\lambda), \tag{3.42}$$

and the linear problem

$$\frac{\mathrm{d}\mathbf{T}_\pm}{\mathrm{d}x} = \left(\frac{\lambda}{2i}\sigma_3 + \mathbf{U}_0(x)\right)\mathbf{T}_\pm, \tag{3.43}$$

where $\mathbf{U}_0(x) \in L_1^{n\times n}(\mathbb{R}_x)$.

In order to obtain the linear problem for $\mathbf{T}_\pm(x, \lambda)$, we differentiate (3.42) with respect to x, and we get

$$\frac{\mathrm{d}\mathbf{T}_+}{\mathrm{d}x}(x, \lambda)\mathbf{T}_+^{-1}(x, \lambda) = \frac{\mathrm{d}\mathbf{T}_-}{\mathrm{d}x}(x, \lambda)\mathbf{T}_-^{-1}(x, \lambda). \tag{3.44}$$

Both sides of (3.44) are entire functions on λ. Now, we shall analyze its asymptotic behavior as $|\lambda| \to \infty$. Since the components of $\mathbf{G}_-(x, \lambda)$ belong to Λ_1^-, the matrix function $\mathbf{T}_-$ has the integral representation

$$\mathbf{T}_-(x, \lambda) = \left(\mathbf{I} + \int_0^\infty \Phi_-(x+s)\mathrm{e}^{-i\lambda s\mathrm{d}s}\right)\mathbf{E}(x, \lambda), \tag{3.45}$$

where $\Phi_-(x+s)$ is absolutely continuous in x, and $\frac{\partial\Phi_-}{\partial x}$, $\frac{\partial\Phi_-}{\partial s}$, $\frac{\partial^2\Phi_-}{\partial x\partial s}$ as function of s belong to $L_1^{2\times 2}(0, \infty)$ [8]. Then, for $\mathrm{Im}\lambda \leq 0$, $\mathbf{T}_-$ has the asymptotic behavior

$$\mathbf{T}_-(x, \lambda) = \left(\mathbf{I} + \frac{\Phi_-(x, 0)}{i\lambda} + o\left(\frac{1}{|\lambda|}\right)\right)\mathbf{E}(x, \lambda), \quad \text{as } |\lambda| \to \infty. \tag{3.46}$$

Therefore, we have

$$\frac{\mathrm{d}\mathbf{T}_-}{\mathrm{d}x}(x, \lambda)\mathbf{T}_-^{-1}(x, \lambda) = \frac{\lambda\sigma_3}{2i} + \frac{1}{2}\left[\sigma_3, \Phi_-(x, 0)\right] + o(1)\text{as } |\lambda| \to \infty. \tag{3.47}$$

Similarly, we have the integral representation for $\mathbf{T}_+(x, \lambda)$

$$\mathbf{T}_+(x, \lambda) = \mathbf{E}^{-1}(x, \lambda)\left(\mathbf{I} + \int_0^\infty \Phi_+(x+s)\mathrm{e}^{i\lambda s\mathrm{d}s}\right), \tag{3.48}$$

and, the asymptotic behavior for $\mathrm{Im}\lambda \geq 0$,

$$\begin{aligned}\frac{\mathrm{d}\mathbf{T}_+}{\mathrm{d}x}(x, \lambda)\mathbf{T}_+^{-1}(x, \lambda) &= \mathbf{T}_+(x, \lambda)\frac{\mathrm{d}\mathbf{T}_+^{-1}}{\mathrm{d}x}(x, \lambda) \\ &= \frac{\lambda\sigma_3}{2i} + \frac{1}{2}\left[\sigma_3, \Phi_+(x, 0)\right] + o(1)\text{as } |\lambda| \to \infty.\end{aligned}$$

Hence, by the Liouville theorem we get

$$\begin{aligned}\frac{\mathrm{d}\mathbf{T}_+}{\mathrm{d}x}(x,\lambda)\mathbf{T}_+^{-1}(x,\lambda) &= \frac{\mathrm{d}\mathbf{T}_-}{\mathrm{d}x}(x,\lambda)\mathbf{T}_-^{-1}(x,\lambda)\\ &= \frac{\lambda\sigma_3}{2i}+\mathbf{U}_0(x),\end{aligned} \tag{3.49}$$

where

$$\mathbf{U}_0(x)=\frac{1}{2}\left[\sigma_3,\Phi_+(x,0)\right]=\frac{1}{2}\left[\sigma_3,\Phi_-(x,0)\right]. \tag{3.50}$$

To conclude, we note that $\mathbf{U}_0$ is anti-diagonal and satisfy involution property, therefore it can be written

$$\mathbf{U}_0=\sqrt{\varkappa}\begin{pmatrix}0 & \overline{\psi}(x)\\ \psi(x) & 0\end{pmatrix}. \tag{3.51}$$

where the parameter $\varkappa$ is positive if $\epsilon=1$ and negative if $\epsilon=-1$.

Step 3. Finally, we prove that

$$\mathbf{M}(\lambda)=\begin{pmatrix}a(\lambda) & \epsilon\bar{b}(\overline{\lambda})\\ b(\lambda) & \overline{a}(\overline{\lambda})\end{pmatrix}$$

is the monodromy matrix for the linear system corresponding to $\psi(x)$. By construction, $\mathbf{T}_+$ are the Jost solution of the linear system associated to ψ. The monodromy matrix is given by

$$\mathbf{M}(\lambda)=\mathbf{T}_+^{-1}\mathbf{T}_1.$$

Using the relations (3.34),(3.40), (3.41), and (3.39), we have

$$\begin{aligned}\mathbf{M}(\lambda) &= \mathbf{E}(-x,\lambda)\mathbf{G}_+(x,\lambda)\mathbf{G}_-(x,\lambda)\mathbf{E}(x,\lambda)\\ &= \mathbf{E}(-x,\lambda)\mathbf{G}(x,\lambda)\mathbf{E}(x,\lambda)\\ &= \mathbf{G}(\lambda)\\ &= \begin{pmatrix}a(\lambda) & \epsilon\bar{b}(\overline{\lambda})\\ b(\lambda) & \overline{a}(\overline{\lambda})\end{pmatrix}.\end{aligned}$$

Formulae for ψ. Now, we describe how to obtain the complex function ψ which determines the matrix coefficient of the linear problem (3.2). We have two cases.

Case $\varkappa>0$. Let

$$\beta(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}b(\lambda)\mathrm{e}^{-i\lambda s}d\lambda \tag{3.52}$$

be the Fourier transform of $b(s)\in\mathcal{S}(\mathbb{R})$. Let $C^+(x,s)$ be the solution of the Winner-Hopf integral equation [1, 8]

$$C^+(x,s)=\beta(s-x)-\int_0^{\infty}\left(\int_x^{\infty}\beta(s-t)\overline{\beta}(s'-t)\mathrm{d}t\right)\cdot C^+(x,s')ds' \tag{3.53}$$

for $s \geq 0$. Then,

$$\psi(x) = \frac{1}{\sqrt{\varkappa}} C^{+}(x, 0). \tag{3.54}$$

Case $\varkappa < 0$. In this case $\psi(x)$ consists of two parts $\psi(x) = \psi^c(x) + \psi^d(x)$.

The first part ψ^c is defined by $b(\lambda)$,

$$\psi^c(x) = \frac{1}{\sqrt{\varkappa}} C^{-}(x, 0) \tag{3.55}$$

where $C^{-}(x, 0)$ is the solution of the Winer-Hopf equation

$$C^{-}(x, s) = \beta(s - x) - \int_0^\infty \left(\int_x^\infty \beta(s - t)\overline{\beta}(s' - t)\mathrm{d}t \right) \cdot C^{-}(x, s')ds' \tag{3.56}$$

The second part ψ^d is determined by the data $(\lambda_i, \gamma_i), b(\lambda)$ in the context of the Riemann problem. In general, when $b \neq 0$ the problem is reduced to the computing the Blashke factor [8]. This way a little bit complicated.

Now, we consider the particular case when

$$b(\lambda) \equiv 0. \tag{3.57}$$

In this case, the function $a(\lambda)$ take the form

$$a(\lambda) = \prod_{j=1}^{n} \frac{\lambda - \lambda_j}{\lambda - \overline{\lambda}_j}, \tag{3.58}$$

and the inverse problem can be solved in closed form.

If $\varkappa > 0$, then $\beta \equiv 0$, and $\psi \equiv 0$.

Now, let us assume $\varkappa < 0$. Let $\{b(\lambda), \lambda_j, \overline{\lambda}_j, \gamma_j, \overline{\gamma}_j\}$ be the spectral data, then the function $\psi \in \mathcal{S}(\mathbb{R})$ is given by

$$\psi(x) = \frac{i}{\sqrt{\varkappa}} \sum_{j=1}^{n} \overline{p}_j(x), \tag{3.59}$$

where, coefficients $p_j(x)$ satisfy the linear system of equations

$$\sum_{k=1}^{n} \left(\frac{1 + \overline{\gamma}_j(x)\gamma_k(x)}{\overline{\lambda}_j - \lambda_k} \right) p_k(x) = \overline{\gamma}_j(x), \quad j = 1, 2, ..., n; \tag{3.60}$$

with

$$\gamma_j(x) = \gamma_j \mathrm{e}^{i\lambda_j x}. \tag{3.61}$$

In order to illustrate the formula (3.59), we derive the case $n = 1$ and $n = 2$.

For $n = 1$, we get

$$\frac{1+|\gamma_1(x)|^2}{-2i\mathrm{Im}(\lambda_1)}p_1(x) = \overline{\gamma}_1(x). \tag{3.62}$$

Solving the last equation for p_1 and substituting in (3.59) we obtain

$$\psi(x) = \frac{2\mathrm{Im}\lambda_1}{\sqrt{\varkappa}} \cdot \frac{\gamma_1(x)}{1+|\gamma_1(x)|^2}, \tag{3.63}$$

where

$$\gamma_1(x) = \gamma_1 e^{i\lambda_1 x}. \tag{3.64}$$

For $n = 2$, we obtain the following algebraic system

$$\begin{aligned}
\left(\frac{1+\overline{\gamma}_1(x)\gamma_1(x)}{\overline{\lambda}_1-\lambda_1}\right)p_1(x) + \left(\frac{1+\overline{\gamma}_1(x)\gamma_2(x)}{\overline{\lambda}_1-\lambda_2}\right)p_2(x) &= \overline{\gamma}_1(x),\\
\left(\frac{1+\overline{\gamma}_2(x)\gamma_1(x)}{\overline{\lambda}_2-\lambda_1}\right)p_1(x) + \left(\frac{1+\overline{\gamma}_2(x)\gamma_2(x)}{\overline{\lambda}_2-\lambda_2}\right)p_2(x) &= \overline{\gamma}_2(x).
\end{aligned}$$

Solving for the unknown functions $p_1(x)$ and $p_2(x)$, we have

$$p_1(x) = \frac{\overline{\gamma}_1(x)\frac{1+\overline{\gamma}_2(x)\gamma_2(x)}{\overline{\lambda}_2-\lambda_2} - \overline{\gamma}_2(x)\frac{1+\overline{\gamma}_1(x)\gamma_2(x)}{\overline{\lambda}_1-\lambda_2}}{\left(\frac{1+\overline{\gamma}_1(x)\gamma_1(x)}{\overline{\lambda}_1-\lambda_1}\right)\left(\frac{1+\overline{\gamma}_2(x)\gamma_2(x)}{\overline{\lambda}_2-\lambda_2}\right) - \left(\frac{1+\overline{\gamma}_2(x)\gamma_1(x)}{\overline{\lambda}_2-\lambda_1}\right)\left(\frac{1+\overline{\gamma}_1(x)\gamma_2(x)}{\overline{\lambda}_1-\lambda_2}\right)}, \tag{3.65}$$

$$p_2(x) = \frac{\overline{\gamma}_1(x)\frac{1+\overline{\gamma}_2(x)\gamma_1(x)}{\overline{\lambda}_2-\lambda_1} - \overline{\gamma}_2(x)\frac{1+\overline{\gamma}_1(x)\gamma_1(x)}{\overline{\lambda}_1-\lambda_1}}{\left(\frac{1+\overline{\gamma}_1(x)\gamma_1(x)}{\overline{\lambda}_1-\lambda_1}\right)\left(\frac{1+\overline{\gamma}_2(x)\gamma_2(x)}{\overline{\lambda}_2-\lambda_2}\right) - \left(\frac{1+\overline{\gamma}_2(x)\gamma_1(x)}{\overline{\lambda}_2-\lambda_1}\right)\left(\frac{1+\overline{\gamma}_1(x)\gamma_2(x)}{\overline{\lambda}_1-\lambda_2}\right)}, \tag{3.66}$$

Finally

$$\psi(x) = i\frac{\overline{p}_1+\overline{p}_2}{\sqrt{\varkappa}}. \tag{3.67}$$

3.4 The inverse Problem for Zero Curvature Equation

Here, we formulate the problem consisting of the reconstruction of solutions of the zero curvature equation in $\mathfrak{sl}(2\mathbb{C})$,

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U},\mathbf{V}] = 0, \tag{3.68}$$

The following results are consequence of Theorems 3.1.2,3.1.1.

Proposition 3.4.1 (Case $\varkappa > 0$). *Let $b(\lambda) \in \mathcal{S}(\mathbb{R}_\lambda)$ be a complex valued function satisfying*

$$|b(\lambda)| < 1.$$

Let us define

$$b(\lambda, t) = e^{-i\lambda^2 t} b(\lambda), \tag{3.69}$$

$$a(\lambda, t) = \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 + |b(\mu, t)|)}{\mu - \lambda - i0} d\mu \right\}. \tag{3.70}$$

for $t \in \mathbb{R}$. *Then, for each* $t \in \mathbb{R}$

$$|a(\lambda, t)|^2 + |b(\lambda, t)|^2 = 1, \quad \textit{for all real } \lambda; \tag{3.71}$$

and there exist an unique $\psi(x, t) \in \mathcal{S}(\mathbb{R}_x)$ *such that*

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda, t) & \bar{b}(\lambda, t) \\ b(\lambda, t) & \bar{a}(\lambda, t) \end{pmatrix}$$

is the monodromy matrix of the linear problem (2.39) associated to $\psi(x, t)$.

Proof. For each fixed, $t \in \mathbb{R}$ we have

$$\begin{aligned} |b(\lambda, t)|^2 &= b(\lambda, t)\bar{b}(\lambda, t) \\ &= e^{-i\lambda^2 t} b(\lambda) e^{i\lambda^2 t} \bar{b}(\lambda) \\ &= e^{-i\lambda^2 t} e^{i\lambda^2 t} b(\lambda)\bar{b}(\lambda) = b(\lambda)\bar{b}(\lambda) \\ &= |b(\lambda)|^2 \\ &< 1. \end{aligned}$$

It is clear that $b(\lambda, t)$ belongs to $\mathcal{S}(\mathbb{R})$, because this space is a ring of functions and both $e^{-i\lambda^2 t}$, $b(\lambda)$ are in $\mathcal{S}(\mathbb{R})$. It follows from Theorem (3.1.1) the existence of the function $\psi(x, t)$ which define a liner problem of type (2.39) for each $t \in \mathbb{R}$ and we get the proposition. □

Proposition 3.4.2 (Case $\varkappa < 0$). *Let* $\{(b(\lambda), \bar{b}(\lambda), \lambda_j, \overline{\lambda}_j, \gamma_j, \overline{\gamma_j})\}$ *be a set consisting of complex number* λ_j, γ_j $(j = 1, 2, ..., n)$ *and* $b(\lambda) \in \mathcal{S}(\mathbb{R}_\lambda)$ *which satisfy the following condition:*

- $\mathrm{Im}\lambda_j > 0$, $\lambda_i \neq \lambda_j$ *if* $i \neq j$,
- $\gamma_j \neq 0$,
- $|b(\lambda)| < 1$.

Let us define

$$b(\lambda, t) = e^{-i\lambda^2 t} b(\lambda), \tag{3.72}$$

$$a(\lambda) = \prod_{j=1}^{n} \frac{\lambda - \lambda_j}{\lambda - \overline{\lambda}_j} \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 + \epsilon|b(\mu, t)|)}{\mu - \lambda - i0} d\mu \right\}. \tag{3.73}$$

for $t \in \mathbb{R}$. Then, for each $t \in \mathbb{R}$

$$|a(\lambda,t)|^2 + |b(\lambda,t)|^2 = 1, \quad \textit{for all real } \lambda; \tag{3.74}$$

and there exist an unique $\psi(x,t) \in \mathcal{S}(\mathbb{R}_x)$ such that

$$\mathbf{M}(\lambda,t) = \begin{pmatrix} a(\lambda,t) & \epsilon\bar{b}(\lambda,t) \\ b(\lambda,t) & \overline{a}(\lambda,t) \end{pmatrix}$$

is the monodromy matrix of the linear problem (2.39) associated to $\psi(x,t)$ and the columns $\left(\mathbf{T}_-^{(1)}(x,t,\lambda), \mathbf{T}_+^{(2)}(x,t,\lambda)\right)$ of the Jost solutions $\mathbf{T}_\pm(x,t,\lambda)$ in (2.91)-(2.92) satisfy the conditions

$$\mathbf{T}_-^{(1)}(x,t,\lambda_j) = \gamma_j(t)\mathbf{T}_+^{(2)}(x,t,\lambda_j), \quad j = 1,2,...,n. \tag{3.75}$$

where

$$\gamma_i(t) = \mathrm{e}^{-i\lambda^2 t}\gamma_i. \tag{3.76}$$

The proof of this statement follows the same arguments as Proposition (3.4.1).Now, we formulate the main result

Theorem 3.4.3. *Let $\psi(x,t)$ be the function defined of the Proposition 3.4.1 or Proposition 3.4.2. The $\mathfrak{sl}(2,\mathbb{C})$-valued functions $\mathbf{U}(x,t,\lambda)$ and $\mathbf{V}(x,t,\lambda)$ defined by*

$$\mathbf{U}(x,t,\lambda) = \mathbf{U}_0(x,t) + \lambda\mathbf{U}_1 \tag{3.77}$$

$$\mathbf{V}(x,t,\lambda) = \mathbf{V}_0(x,t) + \lambda\mathbf{V}_1(x,t) + \lambda^2\mathbf{V}_2(x,t), \tag{3.78}$$

where

$$\mathbf{U}_0(x,t) = \sqrt{\varkappa}\begin{pmatrix} 0 & \overline{\psi}(x,t) \\ \psi(x,t) & 0 \end{pmatrix}, \quad \mathbf{U}_1 = \frac{1}{2i}\sigma_3.$$

$$\mathbf{V}_1(x,t) = -\mathbf{U}_0(x,t), \quad \mathbf{V}_2(x,t) = -\mathbf{U}_1,$$

and

$$\mathbf{V}_0(x,t) = i\sigma_3\mathbf{U}_0^2(x,t) + i\frac{\partial \mathbf{U}_0(x,t)}{\partial x}\sigma_3.$$

satisfy the zero curvature equation (3.68), for any λ.

In the case $b = 0$, we can derive explicit solutions for zero curvature equation.

3.4.1 Proof of Theorem 3.4.3

Using the function $\psi(x,t)$ given in the Proposition 3.4.1 or Proposition 3.4.2 we can construct the matrix $\mathbf{U}(x,t,\lambda)$ (3.77). We only have to prove that the matrix $\mathbf{V}(x,t,\lambda)$ of the form (3.78) satisfy together with $\mathbf{U}$ the zero curvature equation.

Let us define the matrix valued function

$$\mathbf{G}(x,t\lambda) = \mathbf{E}^{-1}(t,\lambda^2)\mathbf{G}(x,t,\lambda)\mathbf{E}(t,\lambda^2), \tag{3.79}$$

where $\mathbf{E}(t,\lambda^2) = \exp\left\{\frac{\lambda^2 t}{2}\sigma_3\right\}$ and $\mathbf{G}(x,t,\lambda)$ given by (3.34).

For each $t \in \mathbb{R}$, the Riemann problem

$$\mathbf{G}(x,t,\lambda) = \mathbf{G}_+(x,t,\lambda)\mathbf{G}_-(x,t,\lambda) \tag{3.80}$$

has a uniquely solution in the Schwartz (Theorem 3.2.4). We define the functions

$$\begin{aligned} \mathbf{T}_+(x,t,\lambda) &= \mathbf{G}_+^{-1}(x,t,\lambda)\mathbf{E}(x,\lambda)\mathbf{E}^{-1}(t,\lambda^2) \\ \mathbf{T}_-(x,t,\lambda) &= \mathbf{G}_-(x,t,\lambda)\mathbf{E}(x,\lambda)\mathbf{E}^{-1}(t,\lambda^2). \end{aligned}$$

For each fixed t, the functions $\mathbf{T}_\pm(x,t,\lambda)$ satisfy the linear problem associated to $\psi(x,t)$ (see the proof of Theorem 3.1.1 and Theorem 3.1.2). Now, we fix $x \in \mathbb{R}$. Since $\mathbf{G}_\pm(x,t,\lambda)$ satisfy the Riemann problem (3.80), we get

$$\mathbf{T}_-(x,t,\lambda) = \mathbf{T}_+(x,t,\lambda)\mathbf{G}(\lambda), \tag{3.81}$$

and from this relation we derive

$$\frac{\partial \mathbf{T}_-}{\partial t}(x,t,\lambda)\mathbf{T}_-^{-1}(x,t,\lambda) = \frac{\partial \mathbf{T}_+}{\partial t}(x,t,\lambda)\mathbf{T}_+^{-1}(x,t,\lambda). \tag{3.82}$$

We recall that the functions $\frac{\partial \mathbf{T}_-}{\partial x}(x,t,\lambda)\mathbf{T}_-^{-1}(x,t,\lambda)$ are non-singular in their respective domain of analyticity. Hence, (3.82) give rise to an entire function of λ. Using the integral representation

$$\mathbf{T}_-(x,t,\lambda) = \left(\mathbf{I} + \int_0^\infty \Phi_-(x,t,s)\mathrm{e}^{-i\lambda s}\mathrm{d}s\right)\mathbf{E}(x,\lambda)\mathbf{E}^{-1}(t,\lambda^2), \tag{3.83}$$

we derive the asymptotic expansion

$$\mathbf{T}_-(x,t,\lambda) = \left(\mathbf{I} + \frac{\Phi_-(x,t,0)}{i\lambda}) - \frac{1}{\lambda^2}\frac{\partial \Phi_-}{\partial s}(x,t,0) + O\left(\frac{1}{|\lambda|^3}\right)\right)\mathbf{E}(x,\lambda)\mathbf{E}^{-1}(t,\lambda^2), \tag{3.84}$$

as $|\lambda| \to \infty$. Differentiating with respect to t, we obtain

$$\frac{\partial \mathbf{T}_-}{\partial t}(x,t,\lambda)\mathbf{T}_-^{-1}(x,t,\lambda) = \mathbf{V}(x,t,\lambda) + O\left(\frac{1}{|\lambda|}\right), \tag{3.85}$$

where

$$\mathbf{V}(x,t,\lambda) = \lambda^2 \mathbf{V}_2 + \lambda \mathbf{V}_1 + \mathbf{V}_0, \tag{3.86}$$

with

$$\mathbf{V}_2(x,t) = \frac{i\sigma_3}{2}, \quad \mathbf{V}_1(x,t) = \frac{1}{2}\left[\Phi_-(x,t,0),\sigma_3\right] = -\mathbf{U}_0(x,t), \tag{3.87}$$

and

$$\mathbf{V}_0(x,t) = \frac{i}{2}\left[\sigma_3, \frac{\partial \Phi_-}{\partial s}(x,t,0)\right] + \frac{i}{2}\left[\Phi_-(x,t,0),\sigma_3\right]\Phi_-(x,t,0). \tag{3.88}$$

On other hand, applying integration by parts successively to (3.83), and differentiating with respect to x we find

$$\frac{\partial \mathbf{T}_-}{\partial x}(x,t,\lambda)\mathbf{T}_-^{-1}(x,t,\lambda) = \frac{\lambda\sigma_3}{2i} + \mathbf{U}_0(x) + \sum_{n=1}^{\infty}\frac{\mathbf{T_n(x,t)}}{(i\lambda)^n} + O\left(|\lambda|^{-\infty}\right) \tag{3.89}$$

as $|\lambda| \to \infty$, $\mathrm{Im}\lambda < 0$. In particular, we have

$$\begin{aligned}\mathbf{T}_1(x,t) &= \frac{1}{2}\left(\left[\sigma_3, \frac{\partial \Phi_-}{\partial s}(x,t,0)\right] + \left[\Phi_-(x,t,0),\sigma_3\right]\Phi_-(x,t,0) + 2\frac{\partial \Phi_-}{\partial x}(x,t,0)\right) \\ &= -i\mathbf{V}_0(x,t) + \frac{\partial \Phi_-}{\partial x}(x,t,0).\end{aligned}$$

However, $\mathbf{T}_-$ satisfies the linear problem corresponding to $\psi(x,t)$ and it follows that

$$\mathbf{T}_n(x,t) = 0 \quad \text{for } n = 1,2,...$$

This implies that

$$\mathbf{V}_0(x,t) = -i\frac{\partial \Phi_-}{\partial x}(x,t,0). \tag{3.90}$$

We can deduce from equation (3.87) that the anti-diagonal part of $\frac{\partial \Phi_-}{\partial x}(x,t,0)$ is equal to $-\frac{\partial U_0}{\partial x}(x,t)\sigma_3$. To find the diagonal part, we consider again that $\mathbf{T}_1(x,t) = 0$. Using the equations (3.87) and (3.90) we get that it is equal to $-\sigma_3\mathbf{U}^2(x,t)$. Finally we obtain

$$\mathbf{V}_0(x,t) = i\sigma_3\mathbf{U}_0^2(x,t) + i\frac{\partial \mathbf{U}_0(x,t)}{\partial x}\sigma_3.$$

Similarly, we can obtain

$$\frac{\partial \mathbf{T}_+}{\partial t}(x,t,\lambda)\mathbf{T}_+^{-1}(x,t,\lambda) = \mathbf{V}(x,t,\lambda) + O\left(\frac{1}{|\lambda|}\right). \tag{3.91}$$

Therefore, $\mathbf{T}_\pm(x,t,\lambda)$ satisfy the linear systems

$$\frac{\partial \mathbf{T}_\pm}{\partial t}(x,t,\lambda) = \mathbf{V}(x,t,\lambda)\mathbf{T}_\pm(x,t,\lambda).$$

Now, recalling that they also are solution of

$$\frac{\partial \mathbf{T}_\pm}{\partial x}(x,t,\lambda) = \mathbf{U}(x,t,\lambda)\mathbf{T}_\pm(x,t,\lambda),$$

it follows that the two linear system are compatible and it implies the zero curvature equation

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U},\mathbf{V}] = 0.$$

3.5 The Nonlinear Schrödinger Equation

The notion of integrable ODE is related with the existence of a number of first integrals. Specially, the integrability property is clear for finite dimensional Hamiltonian systems [4]. In the infinite dimensional case, the situation is more complicated. One of the possible definitions is that [12]: a system of nonlinear differential equations is integrable if it can be represented as the consistence condition of an overdeterminated linear system which is equivalent to zero curvature equation with spectral parameter. An example of an integral system is **the Nonlinear Schrödinger equation (NLS equation)**, a dynamical system generated by the equation

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + 2\varkappa|\psi|^2\psi \quad (\varkappa \in \mathbb{R}) \tag{3.92}$$

with the initial condition

$$\psi(x,t)|_{t=0} = \psi(x). \tag{3.93}$$

As an application of the inverse problem discussed, we construct some solutions of the Nonlinear Schrödinger Equation (NLS equation). This equation arises in various physical contexts, for example, it describes the effects of self-focusing of the envelope of a monochromatic plane wave propagating in nonlinear media [3]. The NLS equation appears also in the theory of surfaces waves on shallow water [5]. Equation (3.92) may be also considered as the Hatree-Fock equation for one dimensional quantum Boson gas equation with point intersection . Physically, the constant $\varkappa$ in (3.92) plays the role of acoupling constant: the case $\varkappa > 0$ corresponds to attractive interaction and $\varkappa < 0$ is the repulsive case. The two cases are essentially different in optical applications, describing self-focusing or defocusing of the light rays [3]. From mathematical point of view, these two cases are also very different because the first one correspond to a selfadjoint linear problem while the second one is related a non-selfadjoint linear problem. The nonlinear Schrödinger equation was first solved by the inverse scattering method by Zakharov and Shabat [18]. In our treatment we shall follow an approach [8], using the result of Chapter 3. In the context of the integrability of NLS equation, the key observation is that, the NLS equation admits a zero curvature representation or Lax pair.

3.6 Zero Curvature Representation for NLS Equation

Consider the zero curvature equation on $\mathfrak{sl}(2,\mathbb{C})$

$$\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{V}}{\partial x} + [\mathbf{U},\mathbf{V}] = 0. \tag{3.94}$$

Suppose that $\mathbf{U}(x,t,\lambda) \in \mathfrak{sl}(2,\mathbb{C})$ is of the form

$$\mathbf{U}(x,t,\lambda) = \mathbf{U}_0(x,t) + \frac{\lambda}{2i}\sigma_3, \quad \mathbf{U}_0(x,t) = \sqrt{\varkappa}\begin{pmatrix} 0 & \overline{\Phi} \\ \Phi & 0 \end{pmatrix} \tag{3.95}$$

where $\Phi = \Phi(x,t)$ is a differentiable complex valued function, $\lambda \in \mathbb{C}$ is a parameter and $\varkappa \in \mathbb{R}$ is a constant. Recall that representation (3.95) comes from a claim of λ-parameter curves in $\mathfrak{sl}(2,\mathbb{C})$ satisfying the involution property (see Chapter 2). Moreover, let us choose $\mathbf{V}(x,t,\lambda) \in \mathfrak{sl}(2,\mathbb{C})$ as follows

$$\mathbf{V} = \mathbf{V}_0 - \lambda \mathbf{U}_0 - \frac{\lambda^2}{2i}\sigma_3, \tag{3.96}$$

$$\mathbf{V}_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \tag{3.97}$$

where α,β and γ are some smooth functions of x and t. It is easy to see that $\mathbf{U}$ and $\mathbf{V}$ in (3.95) and (3.96) satisfy (3.94) for all λ if and only if

$$\frac{\partial \mathbf{U}_0}{\partial t} - \frac{\partial \mathbf{V}_0}{\partial x} + [\mathbf{U}_0,\mathbf{V}_0] = 0, \tag{3.98}$$

$$2i\frac{\partial \mathbf{U}_0}{\partial x} + [\sigma_3,\mathbf{V}_0] = 0. \tag{3.99}$$

The following observation clarify our point of view (3.96) for $\mathbf{V}$.

Proposition 3.6.1. *The differential operators*

$$\mathscr{L} = i\frac{\partial}{\partial x} - i\mathbf{U}_0 + \frac{\lambda}{2}\sigma_3 \tag{3.100}$$

$$\mathscr{M} = \frac{\partial}{\partial t} - \lambda\frac{\partial}{\partial x} - \mathbf{V}_0, \tag{3.101}$$

commute for all λ if and only if $\mathbf{U}_0$ and $\mathbf{V}_0$ satisfy (3.98),(3.99).

The proof is straightforward computation. Thus, the zero curvature equation (3.94) in the class of matrix functions of the form (3.95),(3.96) is reduced to the commutativity condition

$$[\mathscr{L},\mathscr{M}] = 0.$$

Now, we proceed to the study of equations (3.98),(3.99). Putting (3.95),(3.96) into (3.98),(3.99) we get the following set of equations for functions Φ, α, β, γ :

$$
\begin{aligned}
-\frac{\partial\alpha}{\partial x} &+ \sqrt{\varkappa}\,(\overline{\Phi}\gamma - \Phi\beta) = 0, && (3.102)\\
\sqrt{\varkappa}\frac{\partial\overline{\Phi}}{\partial t} &- \frac{\partial\beta}{\partial x} - 2\sqrt{\varkappa}\,\overline{\Phi}\alpha = 0, && (3.103)\\
\sqrt{\varkappa}\frac{\partial\Phi}{\partial t} &- \frac{\partial\gamma}{\partial x} + 2\sqrt{\varkappa}\,\Phi\alpha = 0, && (3.104)\\
i\sqrt{\varkappa}\frac{\partial\overline{\Phi}}{\partial x} &+ \beta = 0, && (3.105)\\
i\sqrt{\varkappa}\frac{\partial\Phi}{\partial x} &- \gamma = 0. && (3.106)
\end{aligned}
$$

From (3.105), (3.106), we have

$$\beta = -i\sqrt{\varkappa}\frac{\partial\overline{\Phi}}{\partial x}, \quad \gamma = i\sqrt{\varkappa}\frac{\partial\Phi}{\partial x}. \tag{3.107}$$

Then, from (3.102) it follows that

$$\frac{\partial\alpha}{\partial x} = i\varkappa\frac{\partial}{\partial x}\left(\overline{\Phi}\Phi\right) \tag{3.108}$$

and hence

$$\alpha = i\varkappa\left(\overline{\Phi}\Phi\right) + \frac{ic}{2}, \tag{3.109}$$

where $c = c(t)$ is an arbitrary function of t.

Putting (3.107), (3.109) into (3.103), (3.104), we observe that the compatibility condition for (3.103) and (3.104) implies that

$$c(t) \in \mathbb{R},$$

and then (3.103) is equivalent to the following equation for Φ:

$$i\frac{\partial\Phi}{\partial t} = -\frac{\partial^2\Phi}{\partial x^2} + 2\varkappa|\Phi|^2\Phi + c(t)\Phi. \tag{3.110}$$

Under the transformation

$$\Phi \mapsto \psi = \Phi\exp\left(i\int c(t)\mathrm{d}t\right),$$

this equation is reduced to NLS equation. We arrive at the following result.

Theorem 3.6.2. *If the matrix valued function* $\mathbf{U}$ *and* $\mathbf{V}$ *of the form (3.95) and (3.96) satisfy the zero curvature equation (3.94), then*

$$\mathbf{U}_0 = \sqrt{\varkappa}\begin{pmatrix} 0 & e^{i\theta}\overline{\psi} \\ e^{-i\theta}\psi & 0 \end{pmatrix}, \tag{3.111}$$

$$\mathbf{V}_0 = \begin{pmatrix} i\left(\varkappa|\psi|^2 + \frac{\theta'}{2}\right) & -\sqrt{\varkappa}e^{i\theta}\frac{\partial\overline{\psi}}{\partial x} \\ \sqrt{\varkappa}e^{-i\theta}\frac{\partial\psi}{\partial x} & -i\left(\varkappa|\psi|^2 + \frac{\theta'}{2}\right) \end{pmatrix}, \tag{3.112}$$

where $\theta = \theta(t)$ *is a real function of* t *and* $\psi = \psi(x,t)$ *is a solution of NLS equation*

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + 2\varkappa|\psi|^2\psi. \tag{3.113}$$

Conversely, an arbitrary real function $\theta(t)$ *and a solution* $\psi(x,t)$ *of NLS equation define a solution* $(\mathbf{U},\mathbf{V})$ *of the zero curvature equation (3.94) by the formulae (3.95), (3.96) and (3.111), (3.112).*

Therefore, the above theorem says that the solutions of the zero curvature equation (3.94) in the class of λ-parametric matrix functions (3.95), (3.96) are parameterized by one real function and a solution of NLS equation.

Corollary 3.6.3. *The nonlinear Schrödinger equation (3.116) for* ψ *is equivalent to the zero curvature equation for the* λ*-parameter matrix functions*

$$\mathbf{U} = \begin{pmatrix} \frac{\lambda}{2i} & \sqrt{\varkappa}\overline{\psi} \\ \sqrt{\varkappa}\psi & \frac{-\lambda}{2i} \end{pmatrix}, \tag{3.114}$$

$$\mathbf{V} = \begin{pmatrix} i\varkappa|\psi|^2 & -i\sqrt{\varkappa}\frac{\partial\overline{\psi}}{\partial x} \\ i\sqrt{\varkappa}\frac{\partial\psi}{\partial x} & -i\varkappa|\psi|^2 \end{pmatrix} - \begin{pmatrix} \frac{\lambda^2}{2i} & \lambda\sqrt{\varkappa}\overline{\psi} \\ \lambda\sqrt{\varkappa}\psi & \frac{-\lambda^2}{2i} \end{pmatrix}. \tag{3.115}$$

Remark 3. *It follows from Proposition (3.6.1) and Corollary (3.6.3) that the NLS equation is equivalent to the commutativity of linear operators* $\mathscr{L}$ *and* $\mathscr{M}$ *(3.100), (3.101) which define the Lax equation pair for the NLS equation [8].*

3.7 Applications of the Inverse Problem to the NLS Equation

Let us consider the dynamical system generated by the nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + 2\varkappa|\psi|^2\psi, \quad (\varkappa \in \mathbb{R}) \tag{3.116}$$

Here, $\psi(x)$ is a given complex valued function belongs to Schwartz space $\mathcal{S}(\mathbb{R})$. We apply the inverse problem and reconstruct the solution $\psi(x,t)$ of the Schrödinger equation from the initial data ψ.

First, we formulate the linear problem associated to ψ,

$$\begin{aligned} \frac{\mathrm{df}}{\mathrm{d}x} &= \mathbf{U}(x,\lambda)\mathbf{f}, \\ \mathbf{f}(x,\lambda)|_{x=y} &= \mathbf{f}^0(y), \end{aligned} \tag{3.117}$$

where

$$\begin{aligned} \mathbf{U}(x,\lambda) &= \mathbf{U}_0(x) + \frac{\lambda}{2i}\sigma_3 \\ &= \sqrt{\varkappa}\begin{pmatrix} 0 & \overline{\psi}(x) \\ \psi(x) & 0 \end{pmatrix} + \begin{pmatrix} \frac{\lambda}{2i} & 0 \\ 0 & -\frac{\lambda}{2i} \end{pmatrix}. \end{aligned}$$

Since $\psi(x) \in \mathcal{S}(\mathbb{R}_x)$, $\mathbf{U} \in L_1^{n\times n}$ and the boundary conditions of system (3.117) are rapidly decreasing functions. Thus, the monodromy matrix $\mathbf{M}(\lambda)$ takes the form

$$\mathbf{M}(\lambda) = \begin{pmatrix} a(\lambda) & \epsilon\overline{b}(\lambda) \\ b(\lambda) & \overline{a}(\lambda) \end{pmatrix} \tag{3.118}$$

where $a(\lambda)$, $b(\lambda) \in \mathcal{S}(\mathbb{R}_\lambda)$ and $|a(\lambda)|^2 - \epsilon|a(\lambda)|^2 = 1$. From the linear problem, we derive the spectral data, proceeding in similar manner as in Section (2.5.1). We have to consider two cases: 1) $\varkappa \geq 0$, where the spectral data consisting only of the function $b(\lambda)$, 2) $\varkappa \geq 0$, where the spectral data is the set $\{b(\lambda), \lambda_j, \gamma_j : j = 1, 2, .., n\}$, λ_j are the zeroes of $a(\lambda)$ on its analytic domain and γ_j are related with the values of the Jost solution in λ_j. If $\varkappa \geq 0$, we define the time dependent function

$$b(t,\lambda) = \mathrm{e}^{-i\lambda^2 t} b(\lambda). \tag{3.119}$$

If $\varkappa < 0$, we also define

$$\gamma_j(t) = \mathrm{e}^{-i\lambda^2 t}\gamma_j \tag{3.120}$$

For each fixed t, the functions $b(\lambda, t)$ and $\gamma_j(t)$ satisfy the conditions of the Theorem 3.1.1 or Theorem 3.1.2. Thus, there exist a function $\psi(x,t)$ such that

$$\mathbf{M}(t,\lambda) = \begin{pmatrix} a(t,\lambda) & \varepsilon\overline{b}(t,\lambda) \\ b(t,\lambda) & \overline{a}(t,\lambda) \end{pmatrix}, \tag{3.121}$$

is the monodromy matrix corresponding to $\psi(x,t)$, with matrix coefficient

$$\mathbf{U}(x,t,\lambda) = \begin{pmatrix} \frac{\lambda}{2i} & \sqrt{\varkappa}\overline{\psi}(x,t) \\ \sqrt{\varkappa}\psi(x,t) & -\frac{\lambda}{2i} \end{pmatrix}. \tag{3.122}$$

Here, we have

$$a(\lambda) = \begin{cases} \prod_{i=1}^{n} \left(\frac{\lambda-\overline{\lambda}_i}{\lambda-\lambda_i}\right) \exp\left(\frac{1}{2\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{\ln(1-|b(\lambda)|^2)}{\mu-\lambda}\mathrm{d}\mu\right) & \text{if } \varepsilon = 1 \\ \exp\left(\frac{1}{2\pi i}\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{\ln(1-|b(\lambda)|^2)}{\mu-\lambda}\mathrm{d}\mu\right) & \text{if } \varepsilon = -1, \end{cases} \tag{3.123}$$

By Theorem 3.4.3, we construct the matrix

$$\mathbf{V}(x,t,\lambda) = \begin{pmatrix} i\varkappa|\psi(x,t)|^2 & -i\sqrt{\varkappa}\frac{\partial\overline{\psi}(x,t)}{\partial x} \\ i\sqrt{\varkappa}\frac{\partial\psi(x,t)}{\partial x} & -i\varkappa|\psi(x,t)|^2 \end{pmatrix} - \begin{pmatrix} \frac{\lambda^2}{2i} & \lambda\sqrt{\varkappa}\overline{\psi}(x,t) \\ \lambda\sqrt{\varkappa}\psi(x,t) & \frac{-\lambda^2}{2i} \end{pmatrix}, \quad (3.124)$$

and $\mathbf{U}$ (3.122) and V (3.124) satisfy the zero curvature equation (3.94) for all λ. Now, It follows from corollary (3.6.3) that $\psi(x,t)$ is a solution of the NLS equation with $\psi(x,0) = \psi(x)$.

Summarizing, for reconstruct the solution $\psi(x,t)$ of NLS equation that satisfy the initial condition $\psi(x)$, we begin with the linear problem corresponding to ψ and derive the spectral data, depending of sign of $\varkappa$. Using these data, we define the time dependent functions (3.119) and (3.120). For each t we apply the inverse problem, and we get the complex valued function $\psi(x,t)$ that is the solution on the NLS equation with the initial condition $\psi(x)$.

$$\psi(x) \quad \xrightarrow{\text{Linear problem}} \quad \begin{cases} b(\lambda) & if\ \varkappa \geq 0 \\ \{b(\lambda), \lambda_j, \gamma_j : j = 1,2,..,n\} & if\ \varkappa \geq 0 \end{cases}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\psi(x,t) \quad \xleftarrow{\text{Inverse problem}} \quad \begin{cases} b(t,\lambda) & if\ \varkappa \geq 0 \\ \{b(t,\lambda), \lambda_j, \gamma_j(t) : j = 1,2,..,n\} & if\ \varkappa \geq 0 \end{cases}$$

Finally, we give some observation about the case $\varkappa = 0$. Firstly, the NLS equation reduce to the linear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2}. \quad (3.125)$$

The behavior of the transition coefficient $a(\lambda)$ and $b(\lambda)$ as $\varkappa \to 0$ is

$$a(\lambda) = 1 + O(|\varkappa|), \quad b(\lambda) = \sqrt{\varkappa}\int_{-\infty}^{\infty} \psi(x)\mathrm{e}^{-i\lambda x}\mathrm{d}x + O(|\varkappa|). \quad (3.126)$$

The integral representation (2.89) for the Jost solution $\mathbf{T}_-(x,\lambda)$ becomes

$$\mathbf{T}_-(x,\lambda) = \mathbf{E}(x,\lambda) + \int_{-\infty}^{x} \mathbf{E}(x-z,\lambda)\mathbf{U}_0(z)\mathbf{E}(z,\lambda)\mathrm{d}z + O(|\varkappa|). \quad (3.127)$$

And the discrete spectrum disappears and the inverse problem reduced to Fourier method [1, 9]. Moreover, the time dynamics of $b(\lambda)$ is given by the Fourier transform of $\psi(x,y)$ subject to ψ. Therefore, if $\varkappa \neq 0$ the inverse problem is interpreted as a nonlinear analogue of the Fourier method.

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